

A hyperbolic problem with non-local constraint describing ion-rearrangement in a model for ion-lithium batteries.

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Abstract

In this paper we study the Fokker-Planck equation arising in a model which describes the charge and discharge process of ion-lithium batteries. In particular we focus our attention on slow reaction regimes with non-negligible entropic effects, which triggers the mass-splitting transition. At first we prove that the problem is globally well-posed. After that we prove a stability result under some hypothesis of improved regularity and a uniqueness result for the stability under some additional condition of the dynamical constraint driving the system.

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1 The model.

The model introduced in [4], to describe the charging and discharging of lithium-ion batteries, governs the evolution of a statistical ensemble of identical particles and is given by the non-local Fokker-Planck equation

$$\tau \partial_t \rho(x, t) = \partial_x \left(v^2 \partial_x \rho(x, t) + (H'(x) - \sigma(t)) \rho(x, t) \right). \quad (\text{FP1})$$

Here H is the free energy of a free particle with thermodynamic state $x \in \mathbb{R}$. The probability density $\rho(\cdot, t)$ describes the state of the whole system at the time t , and σ reflects that the system is subjected to some external forcing. Moreover, $\tau > 0$ is the typical relaxation time of a single particle and $v > 0$ accounts for entropic effect (stochastic fluctuation).

The model (FP1) has two crucial features that cause highly non-trivial dynamics. First, the free energy H is a double-well potential, hence there exist two different stable equilibria for each particle. Second, the system is

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not driven directly but via a time-dependent control parameter, in our case the parameter is the first moment of ρ , that means we impose the following dynamical constraint

$$\int_{\mathbb{R}} x\rho(x, t)dx = \ell(t), \quad (\text{FP2})$$

where ℓ is some given function in time, and a direct calculation shows that (FP2) is equivalent to

$$\sigma(t) = \int_{\mathbb{R}} H'(x)\rho(x, t)dx + \tau \dot{\ell}(t). \quad (\text{FP3})$$

The different relation between ν and τ may cause very different dynamical regimes, which have been studied in [5]. We are going to focus to what are called **slow reaction regimes**, in which we have the coupling

$$\tau = \frac{a}{\log 1/\nu}, \quad (1.1)$$

for some parameter $a \in (0, a_{\text{crit}})$.

It has been seen in [5] that under the assumption (1.1), the solutions of (FP1), (FP2) can be approximated, in the limit $\tau \rightarrow 0$ by means of some simpler problems. In particular, during a suitable range of times, the solutions of (FP1), (FP2) can be approximated by means of the solutions of the problem (MS1) -(MS3) described later. It turns out that during most of the times the function ρ can be approximated by the sum of two Dirac masses. However, during the range of times in which the approximation of (FP1), (FP2) is valid, the mass of ρ is distributed in a region with size x of order one. During those times the mass of ρ is redistributed and, in particular, the mass which is initially localized near the point x_0 is transported to two the neighborhood of two different points, denoted as x_- , x_+ . This redistribution of the mass is described by the model (MS1) -(MS3) and this is the issue considered in this paper. More details concerning the relationship between the problems (FP1), (FP2) and (MS1) -(MS3) are given in [5].

The paper is divided as follows:

- In Section 2 we give some simple assumption on the potential H appearing in the equation (FP1) and we will introduce the problem that we study all along this paper. Moreover we give, in Definition 2.2, some assumptions that the potential H has to satisfy in order that the problem makes sense and is well-posed. After that there is some technical result (namely Lemma 2.3) which determinate some class of potentials which are admissible and compatible with the assumptions in Definition 2.2.
- In Section 3 we prove that there is a unique (in a suitable space) solution of (MS1) -(MS3) in the interval $(-\infty, t_0)$ with $t_0 > -\infty$ as explained heuristically in the paper [5].
- In Section 4 we prove that the problem is globally well posed in all \mathbb{R} , extending the local result proven in Section 3 performed in some interval of the form $[-\infty, t_0]$ to the whole real line.
- In Section 5, the last one, we finally prove that, up to sub-sequences (not relabeled) of re-scaled times $(t_m)_m$ such that $t_m \rightarrow \infty$ the problem converges to some equilibrium. At the end of such section, namely in Subsection 5.1, we prove as well that such equilibrium is unique and independent by the choice of the diverging sequence $(t_m)_m$ as long as the dynamical constraint $\ell(t)$ satisfies some condition in a vicinity of the critical time \tilde{t}_2 in which the mass splitting transition occurs.

2 Introduction to the problem.

2.1 Assumptions on the potential.

In the paper we are going to assume the following hypothesis on the potential H

(A1) H is sufficiently smooth at least $C_{\text{loc}}^3(\mathbb{R})$, and

$$\begin{aligned} H'(x) &= \alpha x + g(x), \\ \|H''\|_{L^\infty(\mathbb{R})} &\leq c < \infty, \end{aligned} \tag{2.1}$$

with $g \in L^\infty$ and $\alpha > 0$.

(A2) There exist constants $x_{**} < x_* < 0 < x^* < x^{**}$ and $\sigma_* < 0 < \sigma^*$ such that

1. $H'(x_*) = H'(x^{**}) = \sigma^*$.
 2. $H'(x^*) = H'(x_{**}) = \sigma_*$.
 3. for each $x \in (x_*, x^*)$ we have that $H'(x) \in (\sigma_*, \sigma^*)$, $H''(x) < 0$ and $H''(x) \geq 0$ for $x \in (x_*, x^*)^c$.
- In particular the inverse of H' has three strictly monotone branches

$$\begin{aligned} X_- &: (-\infty, \sigma^*] \rightarrow (-\infty, x_*], \\ X_0 &: [\sigma_*, \sigma^*] \rightarrow [x_*, x^*], \\ X_+ &: [\sigma_*, \infty) \rightarrow [x_*, \infty). \end{aligned}$$

In what follows we refer to $(x_*, x^*)^c$ as the stable interval, whereas the spinodal region (x_*, x^*) is called the unstable interval. This nomenclature is motivated by the different properties of transport term in (FP1). In both stable intervals adjacent characteristics approach each other exponentially fast, hence there is a strong tendency to concentrate mass into narrow peaks. In the unstable interval, however, the separation of adjacent characteristics de localizes at an exponential rate in time any peak with positive width.

2.2 The mass splitting problem.

Remark 2.1. This is a small remark about notations.

Suppose are given two functions

$$F, G: U \subset \mathbb{R}^n \rightarrow \mathbb{R}.$$

We write

$$F(x_1, \dots, x_n) \lesssim G(x_1, \dots, x_n),$$

if

$$F(x_1, \dots, x_n) \leq C \cdot G(x_1, \dots, x_n),$$

for some constant C which is independent from the variables $(x_1, \dots, x_n) \in U$. ♦

As explained in [5], at the critical time $t_2 \approx \tilde{t}_2$ we expect that the system undergoes a rapid transition from the unstable-stable configuration to a new stable-stable configuration. In order to describe this transition, in particular, to predict the mass distribution between the emerging stable peaks, we are going to study the **mass-splitting model**, which describes the peak widening effect (for a full description about the peak widening model we refer to [5]) on the rescaled time scale $s = (t - \tilde{t}_2)/\tau$ in the limit $\nu \rightarrow 0$. To make the notation simpler we will denote the re-scaled time s simply t . So, all in all the mass splitting problem consists of the following equations

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} ((H'(x) - \sigma(t)) \rho), \tag{MS1}$$

$$\rho(x, t) \sim \frac{m}{\sqrt{\pi} e^{|H''(x_0)|t}} \exp\left(-\frac{(x - x_0)^2}{e^{2|H''(x_0)|t}}\right) + (1 - m)\delta_{x_s}. \tag{MS2}$$

The asymptotics in (MS2) takes place as long as $t \rightarrow -\infty$. The meaning of this asymptotic formula will be explained later, in Definition 2.2.

(MS2) codify the necessity to impose asymptotic initial conditions at $t = -\infty$, which have to reflect the fact that the mass-splitting process starts in an unstable-stable configuration of a two-peaks model, and that the peak on the left is a rescaled Gaussian due to entropic randomness, which means as $t \rightarrow -\infty$, $s \in \{+, -\}$, although in the following we are going to assume $s = +$, the case $s = -$ is simply symmetric, and

$$\int x \rho \, dx = \ell^* \in \mathbb{R}, \tag{MS3}$$

this because the transition happens in a timespan so short that allows us to consider the dynamical constraints $\ell(t) \approx \ell^*$ during the whole transition process.

The mass-splitting problem hence consists of equations (MS1), (MS2) and (MS3).

Moreover, as well as before we require to have well prepared initial data, i.e.

$$a = H''(x_0) < 0, \quad (2.2)$$

$$b = H''(x_+) > 0, \quad (2.3)$$

$$H'(x_0) = H'(x_+) = \sigma_0 \in (\sigma_*, \sigma^*), \quad (2.4)$$

$$m \in (0, 1], \quad (2.5)$$

$$mx_0 + (1 - m)x_+ = \ell^*, \quad (2.6)$$

where indeed (2.2)–(2.4) codify the unstable-stable configuration in which the process starts, and (2.6) is a compatibility condition of the initial data with the dynamical constraint.

In the following it will turn out to be more convenient to use the distribution of ρ instead of the density itself, to this end we define:

$$R(x, t) = \int_{-\infty}^x \rho(y, t) dy. \quad (2.7)$$

Combining (MS1) and (2.7) we obtain the following:

$$\frac{\partial R}{\partial t} = (H'(x) - \sigma) \frac{\partial R}{\partial x}. \quad (2.8)$$

Moreover notice that multiplying (MS1) by x and integrating in the real line, we obtain, after some integration by parts and using (MS3) and (2.7):

$$\sigma(t) = \int H'(x) \frac{\partial R}{\partial x} dx. \quad (2.9)$$

Integrating (MS2) in the interval $(-\infty, x)$ we obtain the formal asymptotic:

$$R(x, t) \sim mQ\left(\frac{x - x_0}{e^{-at}}\right) + (1 - m)\chi_{\{x \geq x_+\}}(x), \quad (2.10)$$

as $t \rightarrow -\infty$. This convergence takes place in L^1 , although it is uniform in compact sets of $\mathbb{R} \setminus \{x_+\}$. In particular, considering a (small) neighbourhood \mathcal{U}_{x_0} of x_0 , $R(x_0 + ye^a t, t) \rightarrow mQ(y)$ as $t \rightarrow -\infty$ in $C_{\text{loc}}^0(\mathcal{U}_{x_0})$. The function Q is given by

$$Q(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^y e^{-\eta^2} d\eta.$$

The aim of this paper is to show, under which conditions the transition of the system gives, at the rescaled time $t = +\infty$, a new stable-stable configuration, i.e. that there exist two non-spinodal states \hat{x}_-, \hat{x}_+ and a numerical value $\tilde{m} \in [-1, 1]$ such that

$$H'(\hat{x}_-) = H'(\hat{x}_+), \quad (m_- - \tilde{m})\hat{x}_- + (m_+ + \tilde{m})\hat{x}_+ = \ell^* = m_-x_- + m_+x_+.$$

The existence and the uniqueness of \tilde{m} is not obvious since the mass-splitting problem involves two sub-tle limits. First one has to show that the asymptotic condition (MS2) gives rise to a well-posed initial value problem at $t = -\infty$. Second, one has to guarantee that solutions do not drift as $t \rightarrow \infty$ along the connected one-parameter family of equilibrium solutions.

We need some general assumption on the function H in order to prove that the problem is well-posed.

Definition 2.2. We say that a function $H \in C_{\text{loc}}^2(\mathbb{R})$ satisfies the **condition H** if, for any function $\phi(t) = \sigma(t) - \sigma_0 \in C^\infty(-\infty, T)$, $T \in \mathbb{R}$ satisfying $|\phi(t)| \leq Me^{-(2a+\delta)t}$, $\delta > 0$ and any $K \in \mathbb{R}$, there exist a unique solution to the ODE problem

$$\begin{cases} \frac{d}{dt}X(t, K) = -(H'(X(t, K)) - \sigma(t)), \\ X(t, K) = x_0 + Ke^{-at} + Y(K, t), \quad \text{as } t \rightarrow -\infty. \end{cases} \quad (2.11)$$

With $Y(K, t) = o(e^{(-a+\delta)t})$.

Moreover for every fixed $t \in (-\infty, T]$ the transformation

$$K \in \mathbb{R} \mapsto X(t, K),$$

is a one-to-one transformation of the real line in a interval $(X_-(t), X_+(t))$ where the functions $X_\pm(t)$ solves the problem

$$\begin{cases} \frac{d}{dt} X_\pm(t) = -(H'(X_\pm(t)) - \sigma(t)), \\ X_\pm(t) = x_\pm + Y_\pm(t), \end{cases} \quad \text{as } t \rightarrow -\infty, \quad (2.12)$$

With $Y_\pm(t) = o(e^{(-a+\delta)t})$.

Where $H'(x_0) = H'(x_\pm) = \sigma_0$. Finally for any fixed $t \in (-\infty, T]$ we have

$$\lim_{K \rightarrow \pm\infty} X(t, K) = X_\pm(t).$$

Definition 2.2 is vacuous if the set of potentials satisfying such condition is empty. For this reason the following lemma gives an example of a class of potentials that satisfy the Condition H.

Lemma 2.3. *Every function $H \in \mathcal{C}_{loc}^3(\mathbb{R})$ such that satisfies the compatibility condition $H'(x_-) = H'(x_0) = H'(x_+) = \sigma_0$ satisfy the conditions in Definition 2.2.*

Proof. We are going to subdivide the proof in several steps

Step 1 We want to show that the solutions of (2.12) are well defined. We are going to do this for the characteristic X_+ , the other case is similar. Let us write X_+ in the following form

$$X_+(t) = x_+ + Y_+(t),$$

where Y_+ is considered to be a perturbation. Than the equation (2.12) reads as

$$Y'_+ = -bY_+ + \phi(t) - \mathcal{O}(Y_+^2),$$

and hence if such a solution exists than it has to take the following form

$$Y_+(t) = \int_{-\infty}^t e^{-b(t-s)} [\phi(s) - \mathcal{O}(Y_+^2(s))] ds.$$

Define the following operator

$$T^+[Y_+](t) = \int_{-\infty}^t e^{-b(t-s)} [\phi(s) - \mathcal{O}(Y_+^2(s))] ds, \quad (2.13)$$

we show that T^+ admits a fixed point in the space

$$J(t_0, \delta) = \left\{ f : |f(t)| \leq e^{(-a+\delta)t}, t < t_0, \delta > 0 \right\},$$

endowed with the norm

$$\|f\|_{J(t_0, \delta)} = \sup_{t < t_0} \left\{ |f(t)| \cdot e^{(a-\delta)t} \right\},$$

for t_0 sufficiently negative and δ small.

To do so take $Y_+ \in J(t_0, \delta)$ and evaluate

$$|T^+[Y_+](t)| = \left| \int_{-\infty}^t e^{-b(t-s)} [\phi(s) - \mathcal{O}(Y_+^2(s))] ds \right| \leq c_1 e^{-(2a+\delta)t} + c_2 e^{2(-a+\delta)t} \leq e^{(-a+\delta)t},$$

for t sufficiently negative, since $c_i = c_i(M, a, b, \delta)$.

At this point take $Y_{+,1}, Y_{+,2} \in J(t_0, \delta)$, and consider

$$|T^+[Y_{+,1}](t) - T^+[Y_{+,2}](t)| \leq \int_{-\infty}^t e^{-b(t-s)} |\mathcal{O}(Y_{+,1}^2(s)) - \mathcal{O}(Y_{+,2}^2(s))| ds. \quad (2.14)$$

In particular we point out that the functions $\mathcal{O}(Y_{+,i}^2(s))$ have an obvious explicit expression, as explained in equation (3.7), in particular, in our case

$$\mathcal{O}(Y_{+,i}^2(s)) = \rho_{+,i}(t) = H'(x_+ + Y_{+,i}(t)) - (H'(x_+) + bY_{+,i}(t)).$$

With this consideration we can rewrite equation (2.14) as

$$\begin{aligned} |T^+[Y_{+,1}](t) - T^+[Y_{+,2}](t)| &\leq \\ &\int_{-\infty}^t e^{-b(t-s)} (|H'(x_+ + Y_{+,1}(s)) - H'(x_+ + Y_{+,2}(s))| + b|Y_{+,1}(s) - Y_{+,2}(s)|) ds. \end{aligned} \quad (2.15)$$

In particular, since $H \in \mathcal{C}_{\text{loc}}^3(\mathbb{R})$ we can say that $H' \in \mathcal{C}_{\text{loc}}^{0,1}(\mathbb{R})$, which means that

$$|H'(x_+ + Y_{+,1}(s)) - H'(x_+ + Y_{+,2}(s))| \leq L|Y_{+,1}(s) - Y_{+,2}(s)|,$$

for some $L > 0$ that depends only on H and on a compact $\mathcal{K} \subset \mathbb{R}$ sufficiently large. With this consideration we can rewrite (2.15) as

$$\begin{aligned} |T^+[Y_{+,1}](t) - T^+[Y_{+,2}](t)| &\leq C(\mathcal{K}, H, b) \int_{-\infty}^t e^{-b(t-s)} |Y_{+,1}(s) - Y_{+,2}(s)| ds \\ &\lesssim \|Y_{0,1} - Y_{0,2}\|_{J(t_0, \delta)} \cdot e^{-bt} \int_{-\infty}^t e^{(b-a+\delta)s} ds \lesssim e^{(-a+\delta)t} \cdot \|Y_{0,1} - Y_{0,2}\|_{J(t_0, \delta)}, \end{aligned}$$

proving that it is a contraction for $t \leq t_0$ and concluding the proof of the first step.

Step 2 We want to prove that for any fixed K there exists a $t_\star = t_\star(K)$ such that (2.11) has a unique solution. To prove this we write

$$X(K, t) = x_0 + Ke^{-at} + Y(K, t),$$

the proof is the same as in the first step only considering $Y(K, t)$ belonging to the following space

$$J(t_0, \delta, K) = \left\{ f \in \mathcal{C}^0 \mid |f(t)| \leq C_K e^{(-a+\delta)t}, t < t_0, \delta > 0 \right\},$$

with ρ defined as in (3.14), with the norm

$$\|f\|_{J(t_0, \delta, K)} = \sup_{t < t_0} \left\{ |f(t)| \cdot e^{(a-\delta)t} \right\}.$$

We point out the fact that, a priori, could happen that $\lim_{K \rightarrow \pm\infty} t_\star(K) = -\infty$.

Step 3 In this step we prove that there exist a time $\tilde{t} = \tilde{t}(\phi, \delta)$ such that the solutions of (2.11) are defined uniformly for all $t \leq \tilde{t}$.

To do so fix a K that we might as well consider to be positive. Since as long as $|X'(\tilde{t}, K)| < \infty$ we can extend the solutions of (2.11) to the right of \tilde{t} , see [2, Theorem 1.1, Chapter 2] for such extensibility theorem, we suppose that there exists a $\hat{t} = \hat{t}(K)$ such that

$$X'(t, K) \xrightarrow{t \rightarrow \hat{t}} \infty. \quad (2.16)$$

For instance $X'(t, K) \xrightarrow{t \rightarrow \hat{t}} +\infty$. The case $X'(t, K) \xrightarrow{t \rightarrow \hat{t}} -\infty$ is similar.

If this happens, then, since $X' = \sigma_0 + \phi(t) - H'(X) \rightarrow +\infty$ then it must be true that

$$-H'(X'(t, K)) \xrightarrow{t \rightarrow \hat{t}} +\infty,$$

and hence considering the structure of H'

$$X'(t, K) \xrightarrow{t \rightarrow \hat{t}} -\infty,$$

which contradicts the hypothesis (2.16).

Step 4 What is left to prove in this last step is that

$$\lim_{K \rightarrow \pm\infty} X(t, K) = X_{\pm}(t).$$

First we claim that for every $t \leq \bar{t}$ there exist a $K > 0$ such that $X'(K, t) > 0$. As usual for $K < 0$ the procedure is similar.

Indeed as we have seen in the second step for every rescaled time t there exists a κ small enough such that the following approximation holds

$$X(\kappa, t) = x_0 + \kappa e^{-at} + o(e^{-at}). \quad (2.17)$$

But then

$$X'(\kappa, t) = \sigma(t) - H'(X(\kappa, t)) = -a\kappa e^{-at} + o(e^{-at}) > 0.$$

Where a is defined in (2.2). Since the approximation in equation (2.17) is valid for every κ as $t \rightarrow -\infty$ we can say that $\lim_{t \rightarrow -\infty} \kappa(t) = +\infty$. Now, considering the flux structure of (2.11), we can argue that $X'(K, t_+) > X'(K, t)$ in some right neighbourhood of t . I want to show that, given an $\eta > 0$ that we are going to consider small, such that $X'(K, t) > \eta$, the system reaches in a finite time the state

$$X'(K, \hat{t}) = \eta, \quad (2.18)$$

where it is not restrictive to set

$$\hat{t} = \inf\{t_+ > t : X'(K, t_+) = \eta\}.$$

We remark the fact that, for every $\eta > 0$ and small there exists a state satisfying (2.18), this because if we suppose $X'(K, \hat{t}) \geq \bar{\eta} > 0$ for every $\hat{t} \geq t$ we would obtain that

$$X(K, \hat{t}) - X(K, t) \geq \bar{\eta}(\hat{t} - t),$$

hence, since $X_+(t)$ is bounded (see Step 1) there would exist a finite time \bar{t} such that $X(K, \bar{t}) = X_+(\bar{t})$, but this contradicts the uniqueness of solution for (2.11).

We want to see that under these assumptions $X(K, s)$ reaches the configuration $X(K, \hat{t})$ in a finite time. Since the characteristics are uniformly bounded as we have seen in the third step we can say that $X(K, \hat{t}) - X(K, t) \leq C_0$, independently of t, \hat{t} . Moreover

$$X(K, \hat{t}) - X(K, t) = \int_t^{\hat{t}} X'(K, z) dz = (\hat{t} - t) X'(K, \bar{t}).$$

Where in the last equality we have used the mean value theorem. Moreover

$$X'(K, \bar{t}) \geq \inf_{\zeta \in [t, \hat{t}]} X'(K, \zeta) > \frac{\eta}{N} > 0,$$

since $X'(K, s) > 0$ in $[t, \hat{t}]$. Whence $(\hat{t} - t) \leq N \frac{C_0}{\eta} < \infty$. Once we have proved that such a state can be reached, in a possibly large, but finite time, and since all this procedure has been done independently by the starting time t we can state that, taking t sufficiently negative there exists a \hat{t} such that

$$H'(X(K, \hat{t})) = \sigma_0 + \phi(\hat{t}) - \eta. \quad (2.19)$$

But $\phi(\hat{t})$ is an $o(e^{-(2a+\delta)\hat{t}})$ function, hence we can assert that for \hat{t} sufficiently negative

$$\sigma_0 - \frac{3}{2}\eta \leq \sigma_0 + \phi(\hat{t}) - \eta \leq \sigma_0 - \frac{\eta}{2},$$

whence

$$\sigma_0 + \phi(\hat{t}) - \eta = \sigma_0 + \mathcal{O}(\eta). \quad (2.20)$$

At this point, since $X(K, \hat{t}) > X(K, t)$, and since the state $X(K, \hat{t})$ has to satisfy (2.19), considering moreover (2.20) we can say that $X(K, \hat{t}) = x_+ + \mathcal{O}(\eta)$, and, thanks to Step 1 we can assert $X_+(\hat{t}) = x_+ + o(1)$, whence

$$|X(K, \hat{t}) - X_+(\hat{t})| \leq \mathcal{O}(\eta) + o(1), \quad (2.21)$$

and indeed the right hand side of (2.21) can be made as small as we want, concluding the proof. \square

3 Well posedness for $t \rightarrow -\infty$.

We want to see that there effectively exist some ρ satisfying (MS1), (MS2), formalized we want to prove the following proposition.

Proposition 3.1. *Suppose H satisfies the conditions in Definition 2.2, and H''' absolutely continuous in compact sets of \mathbb{R} . Suppose also that the following stability condition holds:*

$$(1 - m)a + mb > 0 \quad (3.1)$$

with a and b defined respectively as in (2.2) and (2.3). For any $m \in (0, 1]$ there exist a $T \in \mathbb{R}$ and a unique solution R of (2.8), (2.9), monotonically increasing in x for any $t \in (-\infty, T]$ such that $R(x, t) \in [0, 1]$ and satisfying (2.10), i.e.

$$\|R(\bullet, t) - m\chi_{\{x \geq x_0\}} - (1 - m)\chi_{\{x \geq x_+\}}\|_{L^1(\mathbb{R})} \rightarrow 0, \quad (3.2)$$

as $t \rightarrow -\infty$.

Proof. In order to prove the existence of solutions we define the following class of functions:

$$K_{M,T}(\delta) = \left\{ \sigma \in \mathcal{C}^0 \mid |\sigma(t) - \sigma_0| \leq Me^{-(2a+\delta)t}, \quad t \in (-\infty, T] \right\},$$

for some $\delta > 0$ and sufficiently small.

Our goal is to show that for $T < 0, |T|, M$ sufficiently large it is possible to define a transformation from $K_{M,T}(\delta)$ to itself, whose fixed point is equivalent to solving (2.8), (2.9) and (3.2).

Given $\sigma \in K_{M,T}(\delta)$ and $K \in \mathbb{R}$ let $X(s, K, \sigma) = X(s, K)$ and $X_{\pm}(s, \sigma) = X_{\pm}(s)$ respectively the solutions of (2.11) and (2.12), which are well defined due to the fact that H satisfies the conditions in Definition 2.2. We define then $R(x, s, \sigma)$ as follows:

$$\begin{cases} R(X(t, K, \sigma), t) = mQ(K) + (1 - m)\chi_{\{x \geq X_+(t, \sigma)\}}(X(t, K, \sigma)), \\ R(x, t, \sigma) = 0, & \text{if } x < X_-(t, \sigma), \\ R(x, t, \sigma) = 1, & \text{if } x > X_+(t, \sigma). \end{cases} \quad (3.3)$$

Our goal is to obtain a function $\sigma \in K_{M,T}(\delta)$ such that satisfies equation (2.9), with R defined by means of (3.3). We want to obtain a linearised version of (2.9) where it will be implicitly assumed that $\frac{\partial R}{\partial x}$ can be approximated for $t \rightarrow -\infty$ as two dirac masses at $X_0(t, \sigma) = X(t, 0, \sigma)$ and $X_+(t, \sigma)$ respectively. To this end rewrite (2.9) as:

$$\sigma(t) = H'(X_0(t, \sigma)) \int_{X_-(t, \sigma)}^{X_+(t, \sigma)} \frac{\partial R}{\partial x}(x, t, \sigma) dx + \int_{X_-(t, \sigma)}^{X_+(t, \sigma)} [H'(x) - H'(X_0(t, \sigma))] \frac{\partial R}{\partial x}(x, t, \sigma) dx + (1 - m)H'(X_+(t, \sigma)),$$

whence we obtain

$$\sigma(t) = mH'(X_0(t, \sigma)) + (1 - m)H'(X_+(t, \sigma)) + \int_{X_-(t, \sigma)}^{X_+(t, \sigma)} [H'(x) - H'(X_0(t, \sigma))] \frac{\partial R}{\partial x}(x, t, \sigma) dx. \quad (3.4)$$

At this point, using Taylor expansion we may write

$$\begin{aligned} H'(X_0(t, \sigma)) &= H'(x_0) + H''(x_0)(X_0(t, \sigma) - x_0) + \rho_0(t, \sigma), \\ H'(X_+(t, \sigma)) &= H'(x_+) + H''(x_+)(X_+(t, \sigma) - x_+) + \rho_+(t, \sigma), \end{aligned} \quad (3.5)$$

where indeed ρ_0, ρ_+ are reminder with the obvious explicit expression:

$$\rho_0(t, \sigma) = H'(X_0(t, \sigma)) - (H'(x_0) + H''(x_0)(X_0(t, \sigma) - x_0)), \quad (3.6)$$

$$\rho_+(t, \sigma) = H'(X_+(t, \sigma)) - (H'(x_+) + H''(x_+)(X_+(t, \sigma) - x_+)), \quad (3.7)$$

hence, considering (3.5) into (3.4) we obtain the following expression for $\sigma(t)$:

$$\begin{aligned} \sigma(t) &= [mH'(x_0) + (1 - m)H'(x_+)] + ma(X_0(t, \sigma) - x_0) + (1 - m)b(X_+(t, \sigma) - x_+) + \\ &\quad \int_{X_-(t, \sigma)}^{X_+(t, \sigma)} [H'(x) - H'(X_0(t, \sigma))] \frac{\partial R}{\partial x}(x, t, \sigma) dx + m\rho_0(t, \sigma) + (1 - m)\rho_+(t, \sigma), \end{aligned}$$

with a, b as always defined by means of (2.2), (2.3).

Notice at this point that $[mH'(x_0) + (1-m)H'(x_+)] = \sigma_0$, set $\phi = \sigma - \sigma_0$ and set

$$J(t, \sigma) = \int_{X_-(t, \sigma)}^{X_+(t, \sigma)} [H'(x) - H'(X_0(t, \sigma))] \frac{\partial R}{\partial x}(x, t, \sigma) dx, \quad (3.8)$$

with this consideration we can write equation (3.4) as

$$\phi(t) = ma(X_0(t, \sigma) - x_0) + (1-m)b(X_+(t, \sigma) - x_+) + m\rho_0(t, \sigma) + (1-m)\rho_+(t, \sigma) + J(t, \sigma). \quad (3.9)$$

Our next aim is to express $J(t, \sigma)$ in a more suitable form, to do so integrate by parts equation (3.8) obtaining that

$$J(t, \sigma) = - \int_{X_-(t, \sigma)}^{X_+(t, \sigma)} H''(x) R(x, t, \sigma) dx + R(X_+(t, \sigma), t, \sigma) [H'(X_+(t, \sigma) - H'(X_0(t, \sigma))].$$

Moreover, using the following change of variable

$$\begin{aligned} x &= X(s, K, \sigma), \\ dx &= \frac{\partial X(s, K, \sigma)}{\partial K} dK, \end{aligned}$$

and considering the Condition (H), that ensures that the interval $K \in \mathbb{R}$ is transformed by $X(s, \cdot, \sigma)$ into $(X_-(s, \sigma), X_+(s, \sigma))$ we obtain, using moreover that $R(X_+(t, \sigma), t, \sigma) = m_0$

$$J(t, \sigma) = - \int_{\mathbb{R}} H''(X(t, K, \sigma)) R(X(t, K, \sigma), t, \sigma) \frac{\partial X(s, K, \sigma)}{\partial K} dK + m_0 [H'(X_+(t, \sigma)) - H'(X_0(t, \sigma))].$$

and, using also (3.3)

$$\begin{aligned} J(t, \sigma) &= -m_0 \int_{\mathbb{R}} H''(X(t, K, \sigma)) Q(K) \frac{\partial X(s, K, \sigma)}{\partial K} dK + m_0 [H'(X_+(t, \sigma)) - H'(X_0(t, \sigma))] \\ &= -m_0 \int_{\mathbb{R}} \frac{\partial}{\partial K} (H'(X(t, K, \sigma))) Q(K) dK + m_0 [H'(X_+(t, \sigma)) - H'(X_0(t, \sigma))] \\ &= m_0 \int_{\mathbb{R}} [H'(X(t, K, \sigma)) - H'(X_+(t, \sigma))] Q'(K) dK \\ &\quad + m_0 [H'(X_+(t, \sigma)) - H'(X_0(t, \sigma))]. \end{aligned}$$

Obtaining hence, in the end that

$$J(t, \sigma) = m \int_{\mathbb{R}} (H'(X(t, K, \sigma)) - H'(X_0(t, \sigma))) Q'(K) dK. \quad (3.10)$$

At this point we want to linearise equation (2.11) and (2.12) using Taylor expansion for H' around x_0 and x_+ , obtaining the following equations

$$\frac{d}{dt}(X_0(t, \sigma) - x_0) = -a(X_0(t, \sigma) - x_0) + \phi(t) - \rho_0(t, \sigma), \quad (3.11)$$

$$\frac{d}{dt}(X_+(t, \sigma) - x_+) = -b(X_+(t, \sigma) - x_+) + \phi(t) - \rho_+(t, \sigma), \quad (3.12)$$

$$\frac{d}{dt}X(t, K, \sigma) = -a(X(t, K, \sigma) - x_0) + \phi(t) - \rho(t, K, \sigma), \quad (3.13)$$

with ρ in (3.13) reminder of the form

$$\rho(t, K, \sigma) = [H'(X(t, K, \sigma)) - H'(x_0) - a(X(t, K, \sigma) - x_0)], \quad (3.14)$$

and not a probability density.

We want to linearise also equation (3.10), to do so write first

$$\begin{aligned} (H'(X(t, K, \sigma)) - H'(X_0(t, \sigma))) &= (H'(X(t, K, \sigma)) - \sigma_0) - (H'(X_0(t, \sigma)) - \sigma_0), \\ &= (H'(X(t, K, \sigma)) - H'(x_0)) - (H'(X_0(t, \sigma)) - H'(x_0)), \end{aligned}$$

whence we obtain

$$J(t, \sigma) = ma \int_{\mathbb{R}} (X(t, K, \sigma) - x_0) Q'(K) dK - mb \int_{\mathbb{R}} (X_+(t, \sigma) - x_+) Q'(K) dK + m \int_{\mathbb{R}} \rho(t, K, \sigma) Q'(K) dK - m \int_{\mathbb{R}} \rho_+(t, \sigma) Q'(K) dK, \quad (a)$$

and also, considering that $\int Q'(K) dK = 1$,

$$J(t, \sigma) = ma \int_{\mathbb{R}} (X(t, K, \sigma) - x_0) Q'(K) dK - ma(X_0(t, \sigma) - x_0) + m \int_{\mathbb{R}} \rho(t, K, \sigma) Q'(K) dK - m\rho_0(t, \sigma). \quad (b)$$

We remark the fact that (a) and (b) are different linearisation of $J(t, \sigma)$.

On the other hand we can remove the leading order in the asymptotic of $X(t, K, \sigma) - x_0$ in order to rewrite the problem as a linearised one plus a perturbative term. We write:

$$\begin{aligned} X(t, K, \sigma) - x_0 &= Ke^{-at} + Y(t, K, \sigma), \\ X_0(t, \sigma) - x_0 &= Y_0(t, \sigma), \\ X_+(t, \sigma) - x_+ &= Y_+(t, \sigma). \end{aligned} \quad (3.15)$$

Substituting in (3.9) the results in (3.15) we obtain

$$\phi(t) = m[H''(x_0)Y_0(t, \sigma) + \rho_0(t, \sigma)] + (1 - m)[H''(x_+)Y_+(t, \sigma) + \rho_+(t, \sigma)] + J(t, \sigma).$$

Considering moreover the result in equation (b)

$$\begin{aligned} \phi(t) &= m[H''(x_0)Y_0(t, \sigma) + \rho_0(t, \sigma)] + (1 - m)[H''(x_+)Y_+(t, \sigma) + \rho_+(t, \sigma)] + \\ &\quad ma \int_{\mathbb{R}} Y(t, K, \sigma) Q'(K) dK - maY_0(t, \sigma) + m \int_{\mathbb{R}} \rho(t, K, \sigma) Q'(K) dK - m\rho_0(t, \sigma). \end{aligned}$$

So, at the end, considering that $\int Q' dK = 1$ we obtain

$$\begin{aligned} \phi(t) &= m(aY_0(t, \sigma) + \rho_0(t, \sigma)) + (1 - m)(bY_+(t, \sigma) + \rho_+(t, \sigma)) + \\ &\quad + ma \int_{\mathbb{R}} (Y(t, K, \sigma) - Y_0(t, \sigma)) Q'(K) dK + m \int_{\mathbb{R}} (\rho(t, K, \sigma) - \rho_0(t, \sigma)) Q'(K) dK, \end{aligned} \quad (3.16)$$

and indeed considering (3.15) and linearising using Taylor expansion we obtain:

$$\frac{d}{dt} Y_0(t, \sigma) = -aY_0(t, \sigma) + \phi(t) - \rho_0(t, \sigma), \quad (3.17)$$

$$\frac{d}{dt} Y_+(t, \sigma) = -bY_+(t, \sigma) + \phi(t) - \rho_+(t, \sigma), \quad (3.18)$$

$$\frac{d}{dt} Y(t, K, \sigma) = -aY(t, K, \sigma) + \phi(t) - \rho(t, K, \sigma). \quad (3.19)$$

(3.17)–(3.19) are simply (3.11)–(3.13) with the considerations in (3.15).

At this point we define the following linear operators:

$$\mathcal{U}_0 f(t) = \int_{-\infty}^t e^{-a(t-z)} f(z) dz, \quad \mathcal{U}_+ f(t) = \int_{-\infty}^t e^{-b(t-z)} f(z) dz.$$

Apply the variation of constants method to (3.17)–(3.19), obtaining

$$\begin{aligned} Y_0(t, \sigma) &= \mathcal{U}_0 \phi(t) - \mathcal{U}_0 \rho_0(t, \sigma), \\ Y_+(t, \sigma) &= \mathcal{U}_+ \phi(t) - \mathcal{U}_+ \rho_+(t, \sigma), \\ Y(t, K, \sigma) &= \mathcal{U}_0 \phi(t) - \mathcal{U}_0 \rho(t, K, \sigma). \end{aligned}$$

Using these equation we can transform (3.16) into:

$$\begin{aligned} \phi(t) &= m[a(\mathcal{U}_0 \phi(t) - \mathcal{U}_0 \rho_0(t, \sigma)) + \rho_0(t, \sigma)] + (1 - m)[b(\mathcal{U}_+ \phi(t) - \mathcal{U}_+ \rho_+(t, \sigma)) + \rho_+(t, \sigma)] \\ &\quad + ma \int_{\mathbb{R}} (\mathcal{U}_0 \rho_0(t, \sigma) - \mathcal{U}_0 \rho(t, K, \sigma)) Q'(K) dK + m \int_{\mathbb{R}} (\rho(t, K, \sigma) - \rho_0(t, \sigma)) Q'(K) dK. \end{aligned} \quad (3.20)$$

Rearranging the terms we obtain the following equation:

$$\mathcal{L}\phi(t) = W(t), \quad (3.21)$$

where

$$\mathcal{L}\phi(t) = \phi(t) - ma \int_{-\infty}^t e^{-a(t-z)} \phi(z) dz - (1-m)b \int_{-\infty}^t e^{-b(t-z)} \phi(z) dz, \quad (3.22)$$

and

$$\begin{aligned} W(t) = & m(-a\mathcal{U}_0[\rho_0](t, \sigma) + \rho_0(t, \sigma)) + (1-m)(-b\mathcal{U}_+[\rho_+](t, \sigma) + \rho_+(t, \sigma)) + \\ & + ma \int_{\mathbb{R}} (\mathcal{U}_0[\rho_0](t, \sigma) - \mathcal{U}_0[\rho](t, K, \sigma)) Q'(K) dK + m \int_{\mathbb{R}} (\rho(t, K, \sigma) - \rho_0(t, \sigma)) Q'(K) dK. \end{aligned} \quad (3.23)$$

As it is shown in Lemma A.1 the operator \mathcal{L} defined in (3.22) can be inverted. Moreover, as proved in Lemma A.2, we can explicitly solve the equation

$$\mathcal{L}G(x) = \delta_a(t),$$

whose solution is given in (A.3).

Thanks to the considerations above, using the Green function G we can write the solutions of (3.21) as

$$\phi(t) = \int_{-\infty}^t G(\eta - t) W(\eta) d\eta \quad (3.24)$$

we want to solve this equation using a fixed point argument. To do so recall the space

$$K_{M, t_0}(\delta) = \left\{ \phi \in \mathcal{C}^0 : |\sigma(t) - \sigma_0| = |\phi(t)| \leq M e^{-(2a+\delta)t}, M \in \mathbb{R}, t \in (-\infty, t_0] \right\},$$

Endowed with the norm

$$\|\phi\|_{K_{M, t_0}(\delta)} = \sup_{t \in (-\infty, t_0]} |\phi(t)| \cdot e^{(2a+\delta)t},$$

we want to apply Banach fixed point theorem for the operator

$$T[\phi](t) = \int_{-\infty}^t G(\eta - t) W(\eta) d\eta, \quad (3.25)$$

where the function $W(t) = W(t, \phi)$, i.e. there is a direct non-linear dependence on the function ϕ . To do so we proceed as usual in two steps,

1. Check that T maps $K_{M, T}(\delta)$ to itself for t_0 sufficiently negative.
2. Check that T is a contraction.

Indeed in order to verify 1 and 2 we need some estimate for $|W(t)|$, which are given in Proposition A.3, i.e. equation (A.9) tells us that

$$|W(t)| \leq C e^{-2a t},$$

for some positive $C = C(a, b, m, \delta) < \infty$ uniformly in δ and $t < t_0$ sufficiently negative.

This is the first ingredient in order to prove the fixed point for T . Since $G \in K_{M, t_0}(\delta)$ for each $\delta > 0$, then

$$|T[\phi](t)| = \left| \int_{-\infty}^t G(\eta - t) W(\eta) d\eta \right| \leq C M e^{(2a+\delta)t} \int_{-\infty}^t e^{-(4a+\delta)\eta} d\eta = -\frac{CM}{4a+\delta} e^{-2a t} = -\frac{CM e^{\delta t}}{4a+\delta} e^{-(2a+\delta)t},$$

and indeed

$$\frac{C e^{\delta t}}{4a+\delta} < M,$$

for t sufficiently negative. This proves that the operator T effectively maps $K_{M, t_0}(\delta)$ onto itself if t_0 is sufficiently negative.

To prove that T is indeed a contraction on $K_{M,t_0}(\delta)$ we will have to repeat some calculations which are made explicit in Proposition A.4.

Considering the bound given in (A.10) we can evaluate

$$\begin{aligned} |T[\phi_1](t) - T[\phi_2](t)| &= \left| \int_{-\infty}^t G(\eta - t) [W(\eta, \phi_1) - W(\eta, \phi_2)] d\eta \right| \\ &\leq \int_{-\infty}^t |G(\eta - t)| |W(\eta, \phi_1) - W(\eta, \phi_2)| d\eta \lesssim \|\phi_1 - \phi_2\|_{K_{M,t_0}(\delta)} \int_{-\infty}^t |G(\eta - t)| e^{-(2a+\delta)\eta} d\eta \\ &\stackrel{G \in K_{M,t_0}(\delta)}{\leq} \|\phi_1 - \phi_2\|_{K_{M,t_0}(\delta)} \int_{-\infty}^t e^{-(4a+2\delta)\eta} d\eta \lesssim \|\phi_1 - \phi_2\|_{K_{M,t_0}(\delta)} \cdot e^{-(2a+\delta)t}. \end{aligned} \quad (3.26)$$

Multiplying both sides of (3.26) for $e^{(2a+\delta)t}$, and taking the sup for $t \leq t_0$ we obtain

$$\|T[\phi_1](t) - T[\phi_2](t)\|_{K_{M,t_0}(\delta)} \lesssim e^{-(2a+\delta)t_0} \|\phi_1 - \phi_2\|_{K_{M,t_0}(\delta)},$$

which guarantees that T is a contraction concluding the proof of the theorem. \square

4 Global well posedness.

From now on the re-scaled time t is going to be called t .
The following lemma is the starting point of our analysis.

Lemma 4.1. *Consider σ as defined in (2.9), then, σ is completely determined by the evolution of the characteristics X_{\pm} .*

Proof. Apply integration by parts obtaining

$$\sigma(t) = H'(X_+(t)) - \int_{X_-(t)}^{X_+(t)} H''(x) R(x, t) dx. \quad (4.1)$$

In the same way we can obtain the following equation

$$\sigma(t) = H'(X_-(t)) + \int_{X_-(t)}^{X_+(t)} H''(x) (1 - R(x, t)) dx. \quad (4.2)$$

\square

In view of the simple computations in Lemma 4.1 it is equivalent to prove global existence for σ or for the characteristics X_{\pm} . Recall that

$$\begin{cases} X'_+(t) = -H'(X_+(t)) + \sigma(t), \\ X'_-(t) = -H'(X_-(t)) + \sigma(t), \end{cases}$$

considering (4.1) and (4.2) we obtain the following new differential equations for the characteristics

$$\begin{cases} X'_+(t) = - \int_{X_-(t)}^{X_+(t)} H''(x) R(x, t) dx, \\ X'_-(t) = \int_{X_-(t)}^{X_+(t)} H''(x) (1 - R(x, t)) dx. \end{cases} \quad (4.3)$$

Which is very interesting since we see explicitly in (4.3) that the evolution of X_{\pm} is not influenced by σ , hence we can write (4.3) as $X' = f(X, t)$ with $f: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ and has the following explicit formulation

$$f(x, t) = \left(- \int_{x_2}^{x_1} H''(x) R(x, t) dx, \int_{x_2}^{x_1} H''(x) (1 - R(x, t)) dx \right). \quad (4.4)$$

At this point we can start to study the system (4.3).

Proposition 4.2. *Under the assumptions (2.1) the system (4.3) admits unique solution globally in \mathbb{R} .*

Proof. The function f is Lipschitz continuous and sub-linear in the variable x , hence apply [7, Theorem 2.17] obtaining our global result. \square

The global existence statement proved in Proposition 4.2 gives, thanks to the computations performed in Lemma 4.1 a global existence result also for the function σ , which we showed that the evolution of σ can be completely described in terms of the evolution of the characteristics X_{\pm} .

We would like to refine even further this result, namely we would like to be able to give some L^∞ bound for the characteristics X_{\pm} and, consequently, for the function σ that we might need in the future.

Lemma 4.3. *Define $\Delta X(t) = X_+(t) - X_-(t)$, and consider a potential H which satisfies (2.1). In this case $\Delta X \in L^\infty(\mathbb{R})$.*

Now we can obtain the L^∞ estimates for σ , X_{\pm} .

Proposition 4.4. *Consider a potential H satisfying (2.1), than we have that the function σ defined in (4.1) or equivalently (4.2) and the functions X_{\pm} described by the system (4.3) are not only defined in all \mathbb{R} but they also belong to $L^\infty(\mathbb{R})$.*

Proof. The proof is, at this point, very short. Consider σ as in (4.1), namely

$$\sigma(t) = H'(X_+(t)) - \int_{X_-(t)}^{X_+(t)} H''(x)R(x, t) dx.$$

We know thanks to the previous lemma that $|\Delta X| < L$ hence $\left| \int_{X_-(t)}^{X_+(t)} H''(x)R(x, t) dx \right| \leq \|H\|_{L^\infty} \cdot L < \infty$. Which means that $\int_{X_-(t)}^{X_+(t)} H''(x)R(x, \bullet) dx \in L^\infty(\mathbb{R})$. Considering (2.1), $H'(x) = \alpha x + g(x)$ where g is a L^∞ function. With these considerations we obtain that

$$\sigma(t) = \int H'(x)\rho dx = \alpha \int x\rho dx + \int g(x)\rho dx \leq \alpha \ell^* + \|g\|_{L^\infty(\mathbb{R})} < \infty$$

We've obtained hence that $\sigma, \int_{X_-(t)}^{X_+(t)} H''(x)R(x, \bullet) dx \in L^\infty$, which means, considering (4.1), that $H'(X_+) \in L^\infty$ which implies that $X_+ \in L^\infty$.

To prove that $X_- \in L^\infty$ the reasoning is the same considering σ as in the equation (4.2). \square

5 Stability for $t \rightarrow +\infty$.

Proposition 3.1 assures us that as long as $t \rightarrow -\infty$ the solution to (MS1) stabilize to a convex combination of Dirac- δ measures localized in two points. We expect to recover, after the mass splitting process, a similar configuration.

To do so we are going to show that the system effectively converges to such a form via a standard stability argument in a neighbourhood of $t = +\infty$.

Lemma 5.1. *Define the following function*

$$E_t = \int H(x)\rho dx < \infty, \tag{5.1}$$

where $\rho = \rho(t, x)$ solution of (MS1). Then we have that E_t is decreasing in time.

Proof. Derive equation (5.1), integrate the obtained equation by parts and apply Jensen inequality obtaining that

$$\frac{d}{dt} \left(\int H(x)\rho dx \right) \leq 0$$

Which is exactly what we wanted to prove. \square

At this point it is clear that as $t \rightarrow +\infty$ we have that the system stabilize to some value $E_{+\infty} \leq E_t$ for each $t \in \mathbb{R} \cup \{-\infty\}$, and that (5.1) holds.

Theorem 5.2. *There exists some sequence sequence $(t_m)_{m \in \mathbb{N}}$ such that $t_m \xrightarrow{m \rightarrow \infty} +\infty$. Set $\rho_m(x) = \rho(t_m, x)$, than there exist some x_-, x_0, x_+ , such that $x_{\pm} \in \{H'' \geq 0\}$ and $x_0 \in \{H'' < 0\}$ and*

$$\rho_m(x) \xrightarrow{\star} \sum_{i \in \{-, 0, +\}} m_i \delta_{x_i}(x) \quad (5.2)$$

with $m_- + m_0 + m_+ = 1$ and

$$H'(x_-) = H'(x_+) = H'(x_0) \quad (5.3)$$

Proof. To prove the theorem proceed as follows. Define the **dissipation** of the system as

$$-D(t) = \frac{d}{dt} \left(\int H(x) \rho dx \right) = - \int (H'(x))^2 \rho dx + \left(\int H'(x) \rho dx \right)^2 \leq 0 \quad (5.4)$$

From this definition we obtain that $\int_{-\infty}^T D(t) dt = - \int_{-\infty}^T \frac{d}{dt} E_t dt = E_{-\infty} - E_T < E_{-\infty}$.

At this point, hence, we have obtained that

$$\int_{\mathbb{R}} D(t) dt < \infty, \quad (5.5)$$

which implies that there exists a sequence $(t_m)_m$ such that $D(t_m) \rightarrow 0$ as $m \rightarrow +\infty$. In particular let us select some sequence $t_m \xrightarrow{m \rightarrow \infty} +\infty$, we have that $D(t_m) \rightarrow 0$, i.e.

$$\int (H'(x))^2 \rho_m dx - \left(\int H'(x) \rho_m dx \right)^2 \xrightarrow{m \rightarrow \infty} 0. \quad (5.6)$$

Where $\rho_m(x) = \rho(t_m, x)$.

Moreover the set $(\rho(t, \cdot))_t$ is bounded in $rca(\mathbb{R})$, the set of regular measures, with finite variation. By Banach-Alaoglu theorem is weak- \star compact, hence there exists a (not relabelled) sequence of diverging times $(t_m)_m$ such that $\rho_m \xrightarrow{\star} \rho$.

(5.6) verifies if and only if $H' = k$ for some $k \in \mathbb{R}$ ρ -almost everywhere, for $\rho = \star\text{-}\lim_m \rho_m$. This consideration, together with (2.1), and the fact that H' is strictly monotone in its invertible branches, implies that

$$\rho_m \xrightarrow{\star} \sum_{i \in \{-, 0, +\}} m_i \delta_{x_i},$$

where the points $x_i^+, i \in \{-, 0, +\}$ have to satisfy the condition (5.3) since $H' = k$ ρ -a.s. or, else $\rho_m \xrightarrow{\star} \delta_{\bar{x}}$ for some $\bar{x} \in [x_*, x^*]^c$.

A priori could as well happen that $\rho_m \xrightarrow{\star} \delta_{\infty}$, we want to exclude this eventuality.

To do so consider $\sigma(t_m) = \int H'(x) \rho_m(x) dx \rightarrow \infty$ but $\sigma \in L^{\infty}$ as proved in Proposition 4.4. hence this is absurd. \square

Proposition 5.3. *Set a sequence $(t_m)_m$ such that $\rho_m \xrightarrow{\star} \sum_{i \in \{-, 0, +\}} m_i \delta_{x_i}$. Than for any $f \in \mathcal{C}(D)$ where D is the following compact set of \mathbb{R} : $D = [D_-, D_+]$ with*

$$D_- = \inf_{t \in \mathbb{R}} X_-(t),$$

$$D_+ = \sup_{t \in \mathbb{R}} X_+(t),$$

where X_{\pm} are solution of (4.3), the following equality holds

$$\lim_{m \rightarrow \infty} \int f(x) \rho(t_m, x) dx = \sum_{i \in \{-, 0, +\}} m_i f(x_i).$$

Proof. We point out at first that $|D_{\pm}| < \infty$ thanks the fact that $X_{\pm} \in L^{\infty}$ as shown in Proposition 4.4.

With these considerations, for every time t the mass of the entire system is concentrated in the set $[X_-(t), X_+(t)]$, this has been described in detail in Lemma 2.3, considering also the global result in Proposition 4.2. This implies that the support of the probability density ρ is compact, and independent from the time t , namely the set D . The claim hence follows since, in this setting, $(\mathcal{C}^0(D))^{\star} = rca(D)$ and $\rho(t, \cdot) \in rca(D)$ for all $t \in \mathbb{R}$. \square

Corollary 5.4. Set a sequence $(t_m)_m$ such that $\rho_m \xrightarrow{\star} \sum_{i \in \{-,0,+\}} m_i \delta_{x_i}$. Then

$$\begin{aligned} \lim_{m \rightarrow \infty} \sigma(t_m) &= H'(x_i) & i \in \{-,0,+\} \\ \ell &= \sum_{i \in \{-,0,+\}} m_i x_i \end{aligned}$$

Proof. Apply Proposition 5.3 to the functions $\sigma(t) = \int H'(x) \rho(t, x) dx$ and $\ell^\star = \int x \rho(t, x) dx$ obtaining

$$\sigma(t_m) = \sum_{i \in \{-,0,+\}} m_i H'(x_i) + o(1) = H'(x_i) + o(1). \quad (5.7)$$

and

$$\ell = \int x \rho_m(x) dx = \sum_{i \in \{-,0,+\}} m_i x_i, \quad (5.8)$$

□

Corollary 5.5. For any $t \in \mathbb{R}$ and any $f \in \mathcal{C}(D)$, it is true that $f \in L^1(\mathbb{R}, \rho(t, \cdot))$.

Proof. $\int f(x) \rho(x, t) dx \leq \max_{x \in D} |f(x)| < \infty$. □

Lemma 5.6. Select a sequence $(t_m)_m$ such that $\rho_m \xrightarrow{\star} \sum_{i \in \{-,0,+\}} m_i \delta_{x_i}$. Suppose $x_i \neq x_\star, x^\star$. Define the functions

$$\begin{aligned} m_0(t) &= \int_{x_\star}^{x^\star} \rho(t, x) dx, \\ m_-(t) &= \int_{-\infty}^{x^\star} \rho(t, x) dx, \\ m_+(t) &= \int_{x^\star}^{\infty} \rho(t, x) dx, \end{aligned}$$

then $m_i(t_m) \rightarrow m_i$.

Remark 5.7. Suppose that $x_- = x_0 = x_\star$, hence $\rho_m \xrightarrow{\star} (m_0 + m_-) \delta_{x_\star} + m_+ \delta_{x^\star}$. The lemma above proves that $m_+ = \lim_m m_+(t_m)$, but does not gives any information about the masses $m_-/0$.

Anyway, since $m_- + m_0 + m_+ = 1$ and for every t_m the relation $m_-(t_m) + m_0(t_m) + m_+(t_m) = 1$ still holds, hence we can assert that

$$m_- + m_0 = \lim_m [m_-(t_m) + m_0(t_m)],$$

which is going to suffice for the purposes of our analysis. ♦

After all these considerations we have in particular obtained that

$$\sigma(t_m) \rightarrow H'(x_i),$$

which allows us to choose a triple $(X_-(t_m), X_0(t_m), X_+(t_m))_m$ such that

$$\sigma(t_m) = H'(X_i(t_m)), \quad (5.9)$$

and $X_i(t_m) \rightarrow x_i$ by construction. Considering moreover equation (5.8) and Lemma 5.6 we can hence say that

$$\ell^\star = \sum_{i \in \{-,0,+\}} m_i(t_m) X_i(t_m) + o(1),$$

with $o(1) \rightarrow 0$ as $t_m \rightarrow \infty$.

From now on we are going to refer as X_i to functions that satisfy (5.9), and not any more for solutions of (2.11) with $K = 0, \pm\infty$.

Remark 5.8. a priori we could have different weak- \star limit equilibria for ρ depending on the diverging sequence of times.

In other words, set $(t_m), (t'_m)$ different sequences st $t_m, t'_m \rightarrow +\infty$, set ρ_m, ρ'_m the distributions, the above theorem explains that $\rho_m \xrightarrow{\star} \Sigma, \rho'_m \xrightarrow{\star} \Sigma'$, but might happen, a priori, that $\Sigma \neq \Sigma'$. The next question is: is there uniqueness at the limit? If yes under which conditions? ♦

5.1 The uniqueness problem.

In this section we want to identify some hypothesis under which the problem treated all along this paper has a unique weak- \star limit as $t \rightarrow \infty$.

The first step in our analysis is going to be the following lemma

Lemma 5.9. *Consider D defined as in (5.4). Then $\lim_{t \rightarrow +\infty} D(t) = 0$.*

Proof. Thanks to Corollary 5.5 we can argue that

$$\int (H'(x))^2 \rho(x, t) dx \leq \max_D |H'|^2 < \infty. \quad (5.10)$$

With this consideration in hand, considering that D is defined as in (5.4), it is easy to check that $D, D' \in L^\infty(\mathbb{R})$. Accordingly to (5.4)

$$D(t) = - \underbrace{\left(\int H'(x) \rho(x, t) dx \right)^2}_{=\sigma^2(t) \in L^\infty} + \underbrace{\int (H'(x))^2 \rho(x, t) dx}_{\stackrel{(5.10)}{< \infty}} \in L^\infty.$$

And

$$\begin{aligned} D'(t) &= -2\sigma(t) \cdot \int H'(x) \rho_t(x, t) dx + \int (H'(x))^2 \rho(x, t) dx \stackrel{\text{IbP}+(\text{MS1})}{=} \\ &2\sigma(t) \cdot \int H''(x) (H'(x) - \sigma(t)) \rho(x, t) dx + \int (H'(x))^2 \rho(x, t) dx \leq \\ &2\|\sigma\|_{L^\infty} \left(\max_D |H' \cdot H''| + \|\sigma\|_{L^\infty} \right) + \max_D |H'|^2 < \infty. \end{aligned}$$

Where in the last inequality we applied Corollary 5.5.

At this point, suppose it is not true that $\lim_{t \rightarrow \infty} D(t) = 0$. Hence there exists a sequence such that $D(t_n) > \varepsilon$ for some $\varepsilon > 0$. Define the set $A_\varepsilon = \{t : D(t) \geq \varepsilon\}$, and the set $A_{\varepsilon, n} = A_\varepsilon \cap [n, \infty)$. Moreover $A_{\varepsilon, n} = \bigcup_k A_{\varepsilon, n}^k$, where $A_{\varepsilon, n}^k$ are the connected components of $A_{\varepsilon, n}$, indexed by k . It is true that $\mathcal{L}(A_{\eta, n}) \rightarrow 0$ as $n \rightarrow \infty$, otherwise D wouldn't be $L^1(\mathbb{R}, dt)$ contradicting (5.5), this implies that also $\mathcal{L}(A_{\eta, n}^k) \rightarrow 0$, for every index k . This consideration is valid for every $\eta > 0$, in particular for $\varepsilon/2$.

Select a $t_n \in A_{\varepsilon, n}^k, s_n \notin A_{\varepsilon/2, n}$. These two sequences can be selected in such a way that $|t_n - s_n| \xrightarrow{n \rightarrow \infty} 0$. This is indeed true since $(s_n)_n$ can be chosen such that $|t_n - s_n| \leq 2 \cdot \text{diam}(A_{\varepsilon/2, n}^k) \xrightarrow{n \rightarrow \infty} 0$, inferring via Lagrange theorem to state that there exist a sequence $(\tau_n)_n$ such that $|D(t_n) - D(s_n)| = |D'(\tau_n)| |t_n - s_n|$, but since $t_n \in A_{\varepsilon, n}^k, s_n \notin A_{\varepsilon/2, n}$ is easily obtained that $|D(t_n) - D(s_n)| \geq \frac{\varepsilon}{2}$, but, considering that $|t_n - s_n| \xrightarrow{n \rightarrow \infty} 0$ this would imply that $|D'(\tau_n)| \rightarrow \infty$, contradicting $D' \in L^\infty$ and concluding the proof. \square

This lemma states a very important property, which is that, for every diverging sequence of times we can extract a subsequence such that $\rho_m \xrightarrow{\star} \sum_{i \in \{-, 0, +\}} m_i \delta_{x_i}$.

We would like to understand better the structure of the invertible branches X_i of H .

Lemma 5.10. *Assume the potential H is $C_{loc}^4(\mathbb{R})$, and consider a neighborhood $(\sigma^\star - \delta, \sigma^\star]$, where δ is considered to be small. Recall that, accordingly to (5.9) the functions $X_i, i = -, 0, +$ represent the invertible branches of the potential H . Then the functions $A_{-/0}(\sigma) = X_{-/0}(\sigma) - x_\star$ are uniquely determined, $A_{-/0}(\sigma) = \mathcal{O}(\sqrt{\sigma^\star - \sigma})$ in a vicinity of σ^\star , and, moreover, setting $\sigma^\star - \sigma = \Delta\sigma$ under the regularity assumption made on H , and supposing that $H'''(x_\star) \neq 0$ the following expansion holds*

$$A_{-/0}(\sigma) = \sum_{n=1}^3 a_n^{-/0} (\Delta\sigma)^{n/2} + \mathcal{O}((\Delta\sigma)^2).$$

On the other hand, always in the same vicinity of σ^* , denoting accordingly to the notation introduced, $X_+(\sigma^*) = \sigma^{**}$ we have $X_+(\sigma^*) - x^{**} = \mathcal{O}(\Delta\sigma)$, and

$$X_+(\sigma^*) - x^{**} = \sum_{n=1}^3 a_n^+(\Delta\sigma)^n + \mathcal{O}((\Delta\sigma)^2).$$

Moreover $|A_{-/0}(\sigma)|, |X_+(\sigma^*) - x^{**}| > 0$ if $\sigma \neq \sigma^*$.

Proof. The uniqueness is clear and comes from the definition in (5.9) considering that, by hypothesis, we have been considering potentials H with strictly monotone invertible branches.

By definition of the function X_0 (see (5.9)) we have that $\sigma = H'(X_0(\sigma))$, performing a Taylor expansion of the right hand side of this equation in terms of the perturbation $X_0(\sigma) - x_*$ and considering the fact that $H''(x_*) = 0$ we obtain that

$$\sigma - \sigma^* = \frac{H'''(x_*)}{2} (X_0(\sigma) - x_*)^2 + \mathcal{O}((X_0(\sigma) - x_*)^3). \quad (5.11)$$

Now, from equation (5.11) we can assert that $H'''(x_*) \leq 0$ comparing the signs of the left hand side with the right hand side, moreover, considering that by hypothesis $H'''(x_*) \neq 0$ we obtain that $H'''(x_*) < 0$.

We need a detailed analysis of the factors $A_{-/0}$, where $A_{-/0}(\sigma) = X_{-/0}(\sigma) - x_*$. Let us make the following ansatz

$$A_{-/0}(\sigma) = \sum_{n=1}^3 a_n^{-/0} (\Delta\sigma)^{n/2} + \mathcal{O}((\Delta\sigma)^2), \quad (5.12)$$

which we will justify at the end of this proof, see Remark 5.11, $\Delta\sigma = \sigma^* - \sigma$.

From (5.11), that

$$A_0(\sigma) = \sqrt{\frac{2}{|H'''(x_*)|}} (\sigma^* - \sigma)^{1/2} + o((\sigma^* - \sigma)^{1/2}),$$

The same procedure gives us that

$$A_-(\sigma) = -\sqrt{\frac{2}{|H'''(x_*)|}} (\sigma^* - \sigma)^{1/2} + o((\sigma^* - \sigma)^{1/2}),$$

where indeed we have that $A_-(\sigma) = X_-(\sigma) - x_*$. What is left is to understand the asymptotic behavior of the linear term $X_+(\sigma^*) - x^{**}$, but this is easily obtained performing the same procedure above, in particular we obtain

$$X_+(\sigma) = X_+(\sigma^*) + \frac{\sigma - \sigma^*}{H''(x^{**})} + o(\sigma - \sigma^*), \quad (5.13)$$

hence there is a linear dependence from the parameter σ , since we do know that $H''(X_+(\sigma^*)) > 0$. Equation (5.13) can be justified formally via an argument similar to the one performed in Remark 5.11.

Putting together the results obtained we get

$$\begin{aligned} X_+(\sigma) &= X_+(\sigma^*) - b(\sigma - \sigma^*) + o(\sigma - \sigma^*), \\ X_0(\sigma) &= x_* + c(\sigma^* - \sigma)^{1/2} + o((\sigma^* - \sigma)^{1/2}), \\ X_-(\sigma) &= x_* - c(\sigma^* - \sigma)^{1/2} + o((\sigma^* - \sigma)^{1/2}). \end{aligned} \quad (5.14)$$

Where $c = \sqrt{\frac{2}{|H'''(x_*)|}}$, $b = \frac{1}{H''(x^{**})}$.

At this point the first order expansion is clear. We will need though in the following the expansion of $A_{-/0}$ in (5.12) up to the linear term, i.e. the second order.

To do so we evaluate the next Taylor element in (5.11) we obtain that

$$\Delta\sigma = \frac{1}{2} H'''(x_*) A_0(\sigma)^2 + \frac{1}{6} H^{(4)}(x_*) A_0(\sigma)^3 + \mathcal{O}(A_0(\sigma)^4) = c_1 A_0(\sigma)^2 + c_2 A_0(\sigma)^3 + \mathcal{O}(A_0(\sigma)^4), \quad (5.15)$$

indeed moreover we can express $A_0(\sigma)$ as

$$A_0(\sigma) = a_1^0 (\Delta\sigma)^{1/2} + a_2^0 \Delta\sigma + \mathcal{O}((\Delta\sigma)^{3/2}), \quad (5.16)$$

plugging (5.16) into (5.15) and after some algebraic manipulation we obtain

$$\Delta\sigma = (a_1^0)^2 c_1 \Delta\sigma + \left((a_1^0)^3 c_2 + 2a_1^0 a_2^0 c_1 \right) (\Delta\sigma)^{3/2} + o((\Delta\sigma)^{3/2}),$$

equating the coefficients of $\Delta\sigma$ and $(\Delta\sigma)^{3/2}$ to zero and solving the non-linear system in the unknown a_i^0 we obtain two solutions, namely the two couples (a_1^0, a_2^0) and (a_1^-, a_2^-)

$$a_1^0 = \sqrt{\frac{1}{c_1}}, \quad a_1^- = -\sqrt{\frac{1}{c_1}}, \quad (5.17)$$

$$a_2^0 = -\frac{c_2}{2c_1^2}, \quad a_2^- = -\frac{c_2}{2c_1^2}, \quad (5.18)$$

We recall that c_1, c_2 are defined as in (5.15). □

Remark 5.11. We want to justify equation (5.12). Taylor expansion yields $\Delta\sigma = cA_{-/0}(\sigma)^2 + \mathcal{O}(A_{-/0}(\sigma)^3)$, with $c > 0$. For σ sufficiently close to σ^* both right and left hand side of the equation above are positive, hence it makes sense to take the square root on both sides obtaining the following two equations

$$\begin{aligned} (\Delta\sigma)^{1/2} &= A_0 \sqrt{c + \mathcal{O}(A_0)}, \\ (\Delta\sigma)^{1/2} &= -A_- \sqrt{c + \mathcal{O}(A_-)}. \end{aligned}$$

Define $F^0((\Delta\sigma)^{1/2}, A_0) = A_0 \sqrt{c + \mathcal{O}(A_0)} - (\Delta\sigma)^{1/2}$, a straightforward computation shows that $\frac{\partial F^0}{\partial A_0}((\Delta\sigma)^{1/2}, 0) = \sqrt{c} \neq 0$. By the Implicit function theorem we argue that there exists a function $A_0 = A_0((\Delta\sigma)^{1/2})$ such that $\Delta\sigma = cA_0((\Delta\sigma)^{1/2})^2 + \mathcal{O}(A_0((\Delta\sigma)^{1/2})^3)$, moreover A_0 was a $\mathcal{C}_{\text{loc}}^4(\mathbb{R})$, hence we can express it as $A_0((\Delta\sigma)^{1/2}) = \sum_{n=1}^3 a_n^0 (\Delta\sigma)^{n/2} + \mathcal{O}((\Delta\sigma)^2)$, which is exactly (5.12). A similar approach is valid also for A_- . ◆

At this point we can study the problem of the uniqueness as $t \rightarrow \infty$ for potentials H which are $\mathcal{C}_{\text{loc}}^4(\mathbb{R})$.

Proposition 5.12. *Suppose that the potential H satisfies the hypothesis stated at the beginning of this paper, in Subsection 2.1, and moreover suppose that $\sigma(t) \in (\sigma_* + \eta, \sigma^* - \eta)$ for some $\eta > 0$ and for all times $t > t_0$. Then the limit is unique.*

Proof. First of all we claim that, there exist a local maximum $M_0(t)$ of $\rho(x, t)$, around which all the mass of the unstable region concentrates such that, $\lim_{t \rightarrow \infty} |M_0(t) - X_0(t)| = 0$, where $X_0(t)$ is the only spinodal state such that $\sigma(t) = H'(X_0(t))$. M_0 exists thanks to Theorem 5.2.

Moreover we claim that $M_0 \in \mathcal{C}^1(\mathbb{R})$. This is indeed true since we are considering the mass transported along characteristics, and by Proposition 4.2 these are defined and \mathcal{C}^1 globally in \mathbb{R} .

Since M_0 is a local maximum for ρ satisfies $\rho_x(M_0(t), t) = 0$, and hence, expanding the equation (MS1) into $\rho_t = H''(x)\rho + (H'(x) - \sigma(t))\rho_x$ we obtain that

$$\rho_t(M_0(t), t) = H''(M_0(t))\rho(M_0(t), t). \quad (5.19)$$

Set $\rho(M_0(t), t) = r(t)$. Thanks to the hypothesis $\sigma(t) \in (\sigma_* + \eta, \sigma^* - \eta)$, and since $M_0 = X_0 + o(1)$ we can state that $M_0(t) \in (x_* + \eta', x^* - \eta')$ for some η' small and for t sufficiently large. By the structure of the potential, hence

$$H''(M_0(t)) \leq -\epsilon, \epsilon > 0, \quad (5.20)$$

hence, considering (5.20) we obtain $r'(t) \leq -\epsilon r(t)$. We can, at this point, apply Gronwall inequality to the previous inequality, obtaining $r(t) \leq r(t_0)e^{-\epsilon(t-t_0)} \xrightarrow{t \rightarrow \infty} 0$, Proving that $m_0(t) \rightarrow 0$ as $t \rightarrow \infty$.

Moreover, under these assumptions, $\dot{m}_{\pm} > 0$, hence

$$\ell^* = (m_-^\infty + o(1))X_-(\sigma) + (m_+^\infty + o(1))X_+(\sigma) + o(1)X_0(\sigma), \quad (5.21)$$

where $m_{\pm}^{\infty} = \lim_{t \rightarrow \infty} m_{\pm}(t)$, but, by construction $X_i \in L^{\infty}$, hence equation (5.21) can be restated as

$$\ell^{\star} = m_{-}^{\infty} X_{-}(\sigma) + m_{+}^{\infty} X_{+}(\sigma) + o(1). \quad (5.22)$$

We can, at this point, to prove the uniqueness. Suppose we do not have uniqueness for ρ , than there's no uniqueness for σ either, which means that σ oscillates between two values $\sigma_1 < \sigma_2$, and hence at these two values the convex combination for ℓ^{\star} in (5.22) would give two different results, in particular

$$\ell^{\star} + o(1) = m_{-}^{\infty} X_{-}(\sigma_1) + m_{+}^{\infty} X_{+}(\sigma_1) < m_{-}^{\infty} X_{-}(\sigma_2) + m_{+}^{\infty} X_{+}(\sigma_2) = \ell^{\star} + o(1),$$

which is indeed absurd being ℓ^{\star} a conserved quantity. \square

Remark 5.13. It might be interesting to notice that Proposition 5.12 underlines clearly the equivalence between the vanishing of the spinodal mass and the uniqueness of the weak- \star limit. \blacklozenge

Theorem 5.14. *Under assumptions of Subsection 2.1 on the potential H , if $\ell^{\star} \in [x_{\star}, x^{\star}]^c$, if H is \mathcal{C}^4 around x_{\star} and x^{\star} , and $H^{(4)}(x_{\star}), H^{(4)}(x^{\star}) > 0$, the limit is unique.*

Remark 5.15. In the following proof $o(1)$ is going to be a general perturbation depending only on t such that $o(1) \xrightarrow{t \rightarrow \infty} 0$.

Proof. Define $\overline{\sigma} = \limsup_{t \rightarrow \infty} \sigma(t)$, $\underline{\sigma} = \liminf_{t \rightarrow \infty} \sigma(t)$, and consider the case in which $\ell^{\star} > x^{\star}$. The other case is simply symmetric. Note that this in particular implies that $\underline{\sigma} > \sigma_{\star}$.

We are going to divide the problem in several simple sub-cases.

1. suppose $\underline{\sigma} \geq \sigma^{\star}$. Than $\ell^{\star} = X_{+}(\sigma) + o(1)$, and hence, since ℓ^{\star} has to be constant σ has to converge to a unique limit.
2. the case in which $\overline{\sigma} < \sigma^{\star}$ has already been discussed in detail in Proposition 5.12.
3. suppose $\underline{\sigma} < \sigma^{\star}$ and $\overline{\sigma} > \sigma^{\star}$. This implies that we can choose an arbitrary large time t_{\star} such that $\sigma(t_{\star}) = \sigma^{\star}$ and such that there exist $t_{\star} < t_1 < t_2$ such that $\sigma^{\star} < \sigma_1 = \sigma(t_1) < \sigma_2 = \sigma(t_2) < \overline{\sigma}$.
Let be $x_1 = X_{+}(\sigma_1)$, $x_2 = X_{+}(\sigma_2)$, indeed if $\sigma_1 < \sigma_2$ by strictly monotonicity of H' we can say that $x_1 \neq x_2$. We obtain hence

$$\begin{aligned} \ell^{\star} &= x_1 + o(1), \\ &= x_2 + o(1), \end{aligned}$$

which is absurd.

4. $\sigma_{\star} < \underline{\sigma} < \sigma^{\star}$ and $\overline{\sigma} = \sigma^{\star}$. In this case we are not allowed Gronwall inequality as in Proposition 5.12, but under these assumptions we know that $\dot{m}_{+} \geq 0$, hence there exist

$$\lim_{t \rightarrow \infty} m_{+}(t) = m_{+}^{\infty}. \quad (5.23)$$

We consider some σ close to the extremal value σ^{\star} , $\sigma < \sigma^{\star}$, we are going to perform our analysis on the conserved quantity

$$\ell^{\star} = m_{-}(t) X_{-}(\sigma(t)) + m_0(t) X_0(\sigma(t)) + m_{+}^{\infty} X_{+}(\sigma(t)) + o(1), \quad (5.24)$$

as long as $\sigma \nearrow \sigma^{\star}$. We will be forced to divide the proof in sub-cases.

- (a) Suppose $H'''(x_{\star}) < 0$, whence in this case the asymptotic performed in Lemma 5.10 holds. Inserting (5.14) into (5.24), we obtain

$$\ell^{\star} = m_{+}^{\infty} X_{+}(\sigma^{\star}) + x_{\star} (m_{-}(t) + m_0(t)) + c(m_0(t) - m_{-}(t)) (\Delta\sigma)^{1/2} + \mathcal{O}(\Delta\sigma) + o(1). \quad (5.25)$$

We remark the fact that, thanks to (5.12) the term $\mathcal{O}(\Delta\sigma)$ is $\mathcal{C}_{\text{loc}}^2(\mathbb{R})$.

Now, ℓ^{\star} a constant it has to be independent from $(\Delta\sigma)^{1/2}$, to this end evaluate

$$0 = \frac{\partial \ell^{\star}}{\partial ((\Delta\sigma)^{1/2})} = c(m_0(t) - m_{-}(t)) + \mathcal{O}((\Delta\sigma)^{1/2}). \quad (5.26)$$

From the equation above whence we obtain that

$$m_0(t) = m_-(t) + \mathcal{O}((\Delta\sigma)^{1/2}). \quad (5.27)$$

Moreover

$$m_0(t) + m_-(t) = 1 - m_+(t) = 1 - m_+^\infty + o(1) = B + o(1). \quad (5.28)$$

Considering the result above with the one in equation (5.27) we can argue that

$$m_-(t) = \frac{B + o(1) + \mathcal{O}((\Delta\sigma)^{1/2})}{2} = m_-^\infty + o(1) + \mathcal{O}((\Delta\sigma)^{1/2}). \quad (5.29)$$

Moreover a similar approximation is valid also for m_0 thanks to (5.27), whence

$$m_0(t) = m_0^\infty + o(1) + \mathcal{O}((\Delta\sigma)^{1/2}). \quad (5.30)$$

This approximation is valid as long as the approximation (5.25) holds. Thanks to (5.27) we can say that $m_-^\infty = m_0^\infty$, whence, if σ oscillates, close to the bifurcation point the masses stabilize around the same value.

Consider now the term $c(m_0(t) - m_-(t))$ appearing in (5.25). We know, thanks to Remark 5.16 that

$$c(m_0(t) - m_-(t)) = -2(a_2^{-/0}(1 - m_+^\infty) - m_+^\infty b)(\Delta\sigma)^{1/2} + \mathcal{O}((\Delta\sigma)), \quad (5.31)$$

with $c = |a_1^{-/0}| > 0$. Notice that thanks to (5.18) and the fact that $H^{(4)}(x_\star) > 0$ results that $-2(a_2^{-/0}(1 - m_+^\infty) - m_+^\infty b) > 0$.

At this point, consider an interval $[t_1, t_2]$ such that $\sigma < \sigma^\star$ is decreasing in such interval. This interval always exists since $\sigma \in \mathcal{C}^1$ and $\underline{\sigma} < \sigma^\star$, and such that $\sigma(t_2)$ is close enough such that the approximation (5.31) still holds. Thanks to (5.31) we can argue that in $[t_1, t_2]$, $m_0 - m_-$ is an increasing function. On the other hand as long as $\sigma < \sigma^\star$ we have that $\dot{m}_0 \leq 0$ and $\dot{m}_- \geq 0$, hence $m_0 - m_-$ has to be decreasing according to this consideration. This is indeed an absurd, proving that σ can not oscillate.

- (b) Suppose at last that $H'''(x_\star) = 0$, whence, with the same considerations which have been done before

$$\sigma - \sigma^\star = \frac{H^{(4)}(x_\star)}{6} A_{-/0}^3(\sigma) + \mathcal{O}(A_{-/0}^4(\sigma)).$$

Recall that we are considering $H^{(4)}(x_\star) > 0$. Set $\Delta\sigma$ as in the point a. With a procedure similar to the one performed in Remark 5.11 we can conclude that $A_{-/0}(\sigma) = c(\Delta\sigma)^{1/3} + o((\Delta\sigma)^{1/3})$.

c in the equation above that takes the following explicit expression $c = \left(-\frac{H^{(4)}(x_\star)}{6}\right)^{-1/3} < 0$, whence, for $\sigma < \sigma^\star$ we have that $X_{-/0}(\sigma) - x_\star = c(\Delta\sigma)^{1/3} + o((\Delta\sigma)^{1/3}) < 0$. This indeed implies that $X_{-/0}(\sigma) < x_\star$ for σ sufficiently close to σ^\star , but this contradicts the definition of X_0 , hence we obtained an absurd.

□

Remark 5.16. We want to justify equation (5.31). Define $R(\sigma)$ the $\mathcal{O}((\Delta\sigma))$ function appearing in (5.25). Whence

$$\frac{\partial R}{\partial((\Delta\sigma)^{1/2})} = \mathcal{O}((\Delta\sigma)^{1/2})$$

where $\mathcal{O}((\Delta\sigma)^{1/2})$ is the function appearing in (5.26). Thanks to (5.12) and (5.13) we can express $R(\sigma)$, i.e.

$$R(\sigma) = m_0(t)a_2^0\Delta\sigma + m_-(t)a_2^-\Delta\sigma - m_+^\infty b\Delta\sigma + o(1) + \mathcal{O}((\Delta\sigma)^{3/2}).$$

Recall that, thanks to (5.18) $a_2^0 = a_2^- = a$, and that the term $\mathcal{O}((\Delta\sigma)^{3/2})$ is \mathcal{C}^1 around 0 and that $o(1)$ depends only on t . Using (5.28) we obtain that

$$R(\sigma) = (a(1 - m_+^\infty) - m_+^\infty b)\Delta\sigma + o(1) + \mathcal{O}((\Delta\sigma)^{3/2}).$$

At this point differentiate both sides of the equation above obtaining

$$\frac{\partial}{\partial (\Delta\sigma)^{1/2}} R(\sigma) = 2(a(1 - m_+^\infty) - m_+^\infty b)(\Delta\sigma)^{1/2} + \mathcal{O}(\Delta\sigma),$$

which proves (5.31). ◆

A Estimates for the fixed point theorem and other technicalities.

Lemma A.1. *Consider the operator \mathcal{L} as defined in (3.22), than \mathcal{L} is invertible.*

Proof. Indeed equation (3.22) can be seen as

$$\mathcal{L}\phi(t) = \phi(t) - mK_0 \star \phi(t) - (1 - m)K_+ \star \phi(t), \quad (\text{A.1})$$

for the following convolution kernels

$$\begin{aligned} K_0(t) &= ae^{-at} \chi_{[0, +\infty)}(t), \\ K_+(t) &= be^{-bt} \chi_{[0, +\infty)}(t). \end{aligned}$$

We want to check the invertibility of \mathcal{L} in (3.22).

To begin our analysis consider first the regularity of the left hand side in (3.21). By definition we have that $\phi(t) = \sigma(t) - \sigma_0$ is \mathcal{C}^3 , in fact inherits the same regularity of H around x_0 , in some interval of the form $(-\infty, T]$ for some $T \in \mathbb{R}$. Hence by the convolution structure of the equation (A.1) we have that the left hand side, and hence the right hand side of (3.21) is $\mathcal{C}_{\text{loc}}^3$.

Since (A.1) presents convolutions it seems reasonable to perform some change of variable in such a way to have our functions defined on the positive real line, hence apply the Laplace transform to obtain some information. Setting $z = t - \kappa$ for $\kappa \geq 0$ equation (3.22) reads as

$$\phi(t) - ma \int_0^\infty e^{-a\kappa} \phi(t - \kappa) d\kappa - (1 - m)b \int_0^\infty e^{-b\kappa} \phi(t - \kappa) d\kappa,$$

which is again a convolution equation, and, moreover the convolution kernels didn't change structure. We are performing an asymptotic analysis for t close to $-\infty$, hence it seems reasonable, at least at the moment, to consider t bounded from above by some value t_0 . Hence we can write $t = t_0 - x$ for $x \geq 0$, setting $\phi(t) = \phi(t_0 - x) = \psi(x)$, we can read the above equation as $\psi(x) - ma \int_0^\infty e^{-a\kappa} \psi(x - \kappa) d\kappa - (1 - m)b \int_0^\infty e^{-b\kappa} \psi(x - \kappa) d\kappa$, i.e. $\mathcal{L}\psi(x) = \psi(x) - mK_0 \star \psi(x) - (1 - m)K_+ \star \psi(x)$, which is an equation in convolution form defined on the positive real line.

Performing the substitution $W(t) = W(t_0 - x) = V(x)$, equation (3.21) turn into

$$\mathcal{L}\psi(x) = V(x), \quad (\text{A.2})$$

which has the same regularity of (3.21) but is defined on positive numbers.

We can hence apply Laplace transform on both sides of (A.2) obtaining $L\psi(\theta)(1 - mLK_0(\theta) - (1 - m)LK_+(\theta)) = LV(\theta)$, were $L\psi, LK_0, LK_+$ have respectively the domain

$$\begin{aligned} D(L\psi) &= \{\theta : \text{Re}\theta > a\}, \\ D(LK_0) &= \{\theta : \text{Re}\theta > -a\}, \\ D(L\psi) &= \{\theta : \text{Re}\theta > -b\}. \end{aligned}$$

Hence we can express $L\psi(\theta) = LV(\theta)/C(\theta)$ as a meromorphic function defined on the half complex line $D(LK_0) = \{\theta : \text{Re}\theta > -a\}$. Our aim is to invert the term on the right hand side of this previous equation via inverse Laplace transform.

In particular we want to show that we can express for this particular case the inverse Laplace transform as a residual evaluation, to do so first we have to prove that $|L\psi(\theta)| < M/|\theta|^c$ for some $c > 0$, hence

$$|L\psi(\theta)| = \left| \int_0^\infty e^{-\theta x} \psi(x) dx \right| = \left| \int_0^\infty \left[-\frac{1}{\theta} \frac{d}{dx} e^{-\theta x} \right] \psi(x) dx \right| \stackrel{\text{IBP}}{\leq} \left| -\frac{1}{\theta} e^{-\theta x} \psi(x) \right|_0^\infty + \frac{1}{\theta} \left| \int_0^\infty e^{-\theta x} \psi'(x) dx \right| \leq \frac{c}{\theta}.$$

Where in the last inequality we have proceed as follows. Consider $\frac{d}{dx}\psi(x) = -\frac{d}{dt}\phi(t) = -\frac{d}{dt}\sigma(t)$, and, since $\sigma(t) = \int H'(x)\partial_x R(x, t) dx$ with R defined in (3.3) we obtain

$$\frac{d}{dt}\sigma(t) = H''(X_+(t))(\sigma(t) - H'(X_+)) = [H''(x_+) + \mathcal{O}(X_+(t) - x_+)] \mathcal{O}(X_+(t) - x_+) < c < \infty.$$

Now, we know that ψ is continuous, we have to check were are localized the poles of $L\psi$ i.e. the zeroes of C to understand if we can indeed invert the operator \mathcal{L} .

Set the equation

$$C(\theta) = 1 - \frac{ma}{a+\theta} - \frac{(1-m)b}{b+\theta} = 0.$$

After some algebra we obtain two roots $\theta_1 = 0$, $\theta_2 = -(1-m)a - mb < 0$, which have real part strictly smaller than $-a$, in this way we obtain the following expression for ψ

$$\psi(x) = 2\pi \sum_{i=1,2} \text{Res}_{\theta=\theta_i} \left(e^{\theta x} \frac{LV(\theta)}{C(\theta)} \right),$$

where this last equation is justified by [1, Theorem 16.39].

□

Lemma A.2. *The equation $\mathcal{L}G(t) = \delta_0(t-a)$, is solved for the function $G(t-a)$ where*

$$G(x) = [\delta_0(x) + G_r(x)] \chi_{\mathbb{R}_+}, \quad (\text{A.3})$$

with $G_r(x) = [c_1 + c_2 e^{\theta_2 x}] \chi_{\mathbb{R}_+}(x)$, $c_1 \neq 0, \theta_2 < 0$.

Proof. At first I want to show that every solution of

$$\begin{cases} \mathcal{L}\phi(t) = 0, & \text{for } t \leq t_0, \\ \lim_{t \rightarrow -\infty} \phi(t) = 0, \end{cases} \quad (\text{A.4})$$

satisfies $\phi(t) = 0$ for $t \leq t_0$.

Performing the usual substitution $t = t_0 - x$ we obtain that (A.4) is equivalent to

$$\begin{cases} \mathcal{L}\psi(x) = 0, & \text{for } x \geq 0, \\ \lim_{x \rightarrow \infty} \psi(x) = 0. \end{cases}$$

Applying Laplace transform we obtain that $L\psi(\theta)C(\theta) = 0$ if $D = \{\text{Re}\theta > -a\}$, but C is different from zero in D , which implies that $L\psi$ have to be identically zero in D . This means that $L\psi(\theta) = \int_0^\infty e^{-\theta x} \psi(x) dx = 0$, in D , now we can always see Laplace transform of a function f as Fourier transform of an associated function, up to a constant i.e.

$$Lf(\theta) = \int_0^\infty e^{-\theta x} f(x) dx = \int_{-\infty}^{+\infty} e^{-i\text{Im}\theta x} [\chi_{\mathbb{R}_+}(x) e^{-\text{Re}\theta x} f(x)] dx = \mathcal{F}(f_\theta)(\text{Im}\theta),$$

this is a more flexible way to face the problem. Is easy to check that $\chi_{\mathbb{R}_+}(\bullet) e^{-\text{Re}\theta \bullet} \psi(\bullet) = \psi_\theta \in L^2(\mathbb{R})$, hence as long as $\theta \in D$ we obtain that $\mathcal{F}(\psi_\theta) = 0$ in L^2 . Fourier transform is an invertible operator in L^2 , hence $\psi_\theta = 0$ in L^2 , but since ψ is continuous we obtain that ψ_θ is identically zero and hence $\psi(x) = 0$ for $x \geq 0$.

Consider now the equation

$$\mathcal{L}\phi(t) = \delta_a(t). \quad (\text{A.5})$$

Since $\psi = 0$ for $t < a$, with the substitutions

$$\begin{aligned} t - a &= x, \\ \zeta + a &= z, \\ \phi(t) &= \varphi(t-a) = \varphi(x). \end{aligned}$$

Equation (A.5) turns into the following

$$\varphi(x) - ma \int_0^x e^{-a(x-\zeta)} \varphi(\zeta) d\zeta - (1-m)b \int_0^x e^{-b(x-\zeta)} \varphi(\zeta) d\zeta = \delta_0(x). \quad (\text{A.6})$$

A straightforward application of transform methods may not be efficient, given the strong discontinuity presented in the problem given by the function δ_0 , in this spirit we try to substitute φ with a suitable decomposition that may lead to a problem sufficiently regular to apply the Laplace transform.

To do so, consider φ as $\varphi = \delta_0 + G_r$. Inserting this function in such a form equation (A.6) reads as:

$$\mathcal{L}G_r(x) = ma e^{-ax} + (1-m)b e^{-bx},$$

where the member on the right hand side is suitable for application of transform methods. Applying Laplace transform to both sides we obtain the following equation defined in the half complex plane $\{\text{Re}\theta > -a\}$

$$\left(1 - \frac{ma}{\theta+a} - \frac{(1-m)b}{\theta+b}\right) LG_r(\theta) = \frac{ma}{\theta+a} + \frac{(1-m)b}{\theta+b},$$

which is equivalent, after some algebraic manipulation to the following

$$LG_r(\theta) = \frac{\theta(ma + (1-m)b) + ab}{\theta^2 + ((1-m)a + mb)\theta} = \frac{B(\theta)}{C(\theta)},$$

We have hence obtained that LG_r can be expressed as a meromorphic function which has 2 simple poles located at $\theta_1 = 0, \theta_2 < 0$. Heaviside inversion theorem (see, for instance, [6] for a proof of such), can be applied, expressing G_r as

$$G_r(x) = \frac{B(\theta_1)}{C'(\theta_1)} e^{\theta_1 x} + \frac{B(\theta_2)}{C'(\theta_2)} e^{\theta_2 x} = c_1 + c_2 e^{\theta_2 x}.$$

We've obtained that if $G(x)$ is solution to (A.5) we have that $G(x) = [\delta_0(x) + G_r(x)] \chi_{\mathbb{R}_+}$ with $G_r(x) = 0$ as long as $x < 0$ and $G_r \in L^\infty(\mathbb{R})$ since $\theta_2 < 0$.

□

In the proof of Proposition 3.1 we use a fixed point argument, i.e. we want to show that the operator T defined by mean of formula (3.25) is indeed a contraction between Banach spaces. To do so we require the following estimates.

Proposition A.3. *Let be W be defined by means of (3.23), than we have that there exists some positive $C = C(a, b, m, \delta) < \infty$ uniformly in δ , and $t < t_0$ sufficiently negative such that*

$$|W(t)| \leq C e^{-2at} \quad (\text{A.7})$$

Proof. Recall that W is defined as follows

$$\begin{aligned} W(t) = & m(-a\mathcal{U}_0[\rho_0](t, \sigma) + \rho_0(t, \sigma)) + (1-m)(-b\mathcal{U}_+[\rho_+](t, \sigma) + \rho_+(t, \sigma)) + \\ & + ma \int_{\mathbb{R}} (\mathcal{U}_0[\rho_0](t, \sigma) - \mathcal{U}_0[\rho](t, K, \sigma)) Q'(K) dK + m \int_{\mathbb{R}} (\rho(t, K, \sigma) - \rho_0(t, \sigma)) Q'(K) dK \end{aligned} \quad (\text{3.23})$$

Recall as well that

$$\begin{aligned} \rho_0 &= \mathcal{O}(Y_0^2) \\ \rho_+ &= \mathcal{O}(Y_+^2) \\ \rho &= \mathcal{O}\left((Ke^{-at} + Y)^2\right) \end{aligned} \quad (\text{A.8})$$

and $Y_0, Y_+, Y = o(1)$ as $t \rightarrow -\infty$.

Recall as well, as shown in Lemma 2.3 that $|Y_{0/+}(t)| \leq e^{(-a+\delta)t}$, hence $|\rho_{0/+}(t)| \leq C_H e^{2(-a+\delta)t}$. A straightforward computation shows moreover that $|\mathcal{U}_0[\rho_{0/+}](t, \sigma)| \leq \tilde{C}_H e^{2(-a+\delta)t}$. By the definition of $\rho(K, t)$ we know that $\rho(K, t) = \mathcal{O}((X(K, t) - x_0)^2)$, and, since $\lim_{K \rightarrow \pm\infty} |X(K, t)| = |X_{\pm}(t)| < \infty$, we can say that the function $X(\cdot, t)$ is bounded as long as the extremal characteristics X_{\pm} exist. In particular since $X_{\pm} = x_{\pm} + Y_{\pm}$ with $Y_{\pm}(t) \leq e^{(-a+\delta)t}$ we can say that

$$\begin{aligned} X_+(t) &\leq x_+ + \frac{1}{2}, \\ X_-(t) &\geq x_- - \frac{1}{2}, \end{aligned}$$

hence we can bound uniformly $|X(K, t)| \leq x_+ - x_- + 1$, which implies that $\left| \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \rho(K, t) e^{-K^2} dK \right| \leq \bar{C}_H e^{2(-a+\delta)t}$. We point out the fact that $Q'(K) = \frac{1}{\sqrt{\pi}} e^{-K^2}$.

A straightforward computation as before shows hence that $\left| \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathcal{U}_0[\rho](K, t) e^{-K^2} dK \right| \leq \bar{C}_H e^{2(-a+\delta)t}$. At this point, considering W as in equation (3.23) we can hence assert that

$$|W(t)| \leq C e^{2(-a+\delta)t} \quad (\text{A.9})$$

for some positive $C = C(a, b, m, \delta, H) < \infty$ uniformly in δ and $t < t_0$ sufficiently negative. \square

Proposition A.4. *Let W be defined via equation (3.23), and let $\phi_1, \phi_2 \in K_{M, t_0}(\delta)$. Consider H with the same properties as in Lemma 2.3. We recall again that all the functions $\rho_i, Y_i, i = 0, +$ have a direct dependence on the parameter σ , which is equivalent as having a dependence for a parameter $\phi \in K_{M, t_0}(\delta)$, then*

$$|W(t, \phi_1) - W(t, \phi_2)| \lesssim e^{-(2a+\delta)t} \cdot \|\phi_1 - \phi_2\|_{K_{M, t_0}(\delta)}. \quad (\text{A.10})$$

Proof. The functions $\rho_i, Y_i, i = 0, +$ have a direct dependence on $\phi \in K_{M, t_0}(\delta)$, in particular the dependence is given by

$$Y_{0/+} = \mathcal{U}_{0/+}[\phi] - \mathcal{U}_{0/+}[\rho_{0/+}], \quad (\text{A.11})$$

$$Y = \mathcal{U}_0[\phi] - \mathcal{U}_0[\rho], \quad (\text{A.12})$$

and the functions ρ_i are defined as in (A.8).

We want to derive first some estimate for the element $|\mathcal{U}_0(\phi_1 - \phi_2)(t)|$, with $\mathcal{U}_0[f](t) = \int_{-\infty}^t e^{-a(t-s)} f(s) ds$, hence

$$\begin{aligned} |\mathcal{U}_0(\phi_1 - \phi_2)(t)| &= \left| \int_{-\infty}^t e^{-a(t-z)} (\phi_1(z) - \phi_2(z)) dz \right| \\ &\leq \|\phi_1 - \phi_2\|_{K_{M, t_0}(\delta)} \cdot e^{-at} \int_{-\infty}^t e^{-(a+\delta)z} dz \leq \frac{1}{-(a+\delta)} e^{-(2a+\delta)t} \cdot \|\phi_1 - \phi_2\|_{K_{M, t_0}(\delta)}, \end{aligned} \quad (\text{A.13})$$

this estimate will be useful in the following.

Since H satisfies the same properties as the H considered in Lemma 2.3 we can say that, for t_0 sufficiently negative, $\rho_0(t, \phi_i) = R(Y_0(t, \phi_i))$, where we decide to express $R(Y_0(t, \phi_i))$ in its integral form, i.e.

$$R(Y_0(t, \phi_i)) = \int_0^{Y_0(t, \phi_i)} \frac{H'''(x_0 + \zeta)}{2} (Y_0(t, \phi_i) - \zeta)^2 d\zeta,$$

since it gives a better idea of the regularity of the reminder. We can express the reminder in such a way since by our hypothesis in Proposition 3.1 H''' is locally absolutely continuous. Since Y_0 is at least \mathcal{C}^1 (see Lemma 2.3), then $R(Y_0)$ is \mathcal{C}^1 as well.

Consider at this point the function $R(x) = \int_0^x \frac{H'''(x_0 + \zeta)}{2} (x - \zeta)^2 d\zeta$. $R \in \mathcal{C}_{\text{loc}}^{0,1} \cup \mathcal{C}_{\text{loc}}^1$, hence for x, y in a compact set \mathcal{K} $|R(x) - R(y)| \leq L(\mathcal{K})|x - y|$. We claim that

$$\lim_{\mathcal{L}(\mathcal{K}) \rightarrow 0, 0 \in \mathcal{K}} L(\mathcal{K}) = 0. \quad (\text{A.14})$$

where \mathcal{L} is the Lebesgue measure on \mathbb{R} .

Indeed, for locally differentiable functions g on compact sets \mathcal{K} , we have that $\|g\|_{\mathcal{C}_{\mathcal{K}}^{0,1}} \leq \sup_{x \in \mathcal{K}} |g'(x)|$, which

means we can bound the Lipschitz norm in term of the sup of the derivative in the compact set. This means that, if we prove that $R'(x) \rightarrow 0$ as $|x| \rightarrow 0$ we prove (A.14).

A computation shows that $R'(x) = x \int_0^x H'''(x_0 + z) dz - \int_0^x z H'''(x_0 + z) dz \xrightarrow{x \rightarrow 0} 0$, since $H \in \mathcal{C}_{\text{loc}}^3$.

Now, $R(Y_0(t, \phi_i)) = \rho_0(t, \phi_i)$, this means we have obtained that

$$|\rho_0(t, \phi_1) - \rho_0(t, \phi_2)| \leq L(t) \cdot |Y_0(t, \phi_1) - Y_0(t, \phi_2)|, \quad (\text{A.15})$$

with $L(t) = o(1)$ as $t \rightarrow -\infty$.

Moreover, since Y_0 is defined via equation (A.11), we can infer that

$$|Y_0(t, \phi_1) - Y_0(t, \phi_2)| \leq |\mathcal{U}_0(\phi_1 - \phi_2)(t)| + |\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)|,$$

obtaining,

$$|\rho_0(t, \phi_1) - \rho_0(t, \phi_2)| \leq L(t) \left(|\mathcal{U}_0(\phi_1 - \phi_2)(t)| + |\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| \right).$$

Hence, considering the term $-a\mathcal{U}_0[\rho_0(\phi)](t) + \rho_0(t, \phi)$ in (3.23), with these estimates, we can argue that

$$\begin{aligned} & |-a(\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t) + \rho_0(t, \phi_1) - \rho_0(t, \phi_2))| \\ & \leq L(t) |\mathcal{U}_0(\phi_1 - \phi_2)(t)| + (-a + L(t)) |\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)|, \end{aligned} \quad (\text{A.16})$$

hence what's left to understand is how to bound the term $|\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)|$.

Repeating the procedure in equation (A.15) we can argue that, there exists a $0 < q < 1$ that we can make as small as we want since $q = \sup_{t \leq t_0} \{L(t)\}$, such that,

$$\begin{aligned} |\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| & \leq q |\mathcal{U}_0(Y_0(t, \phi_1) - Y_0(t, \phi_2))| \\ & \stackrel{(\text{A.11})}{=} q |\mathcal{U}_0^2(\phi_1 - \phi_2)(t) - \mathcal{U}_0^2(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| \\ & \leq \sum_{n=1}^N q^n |\mathcal{U}_0^{n+1}(\phi_1 - \phi_2)(t)| + q^N |\mathcal{U}_0^{N+1}(\rho_0(\phi_1) - \rho_0(\phi_2))(t)|. \end{aligned}$$

Is easy to check that

$$|\mathcal{U}_0^{N+1}(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| \leq \left(\frac{1}{-a + 2\delta} \right)^{N+1} e^{2(-a+\delta)t},$$

which in turn implies the following

$$q^N |\mathcal{U}_0^{N+1}(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| \leq \frac{1}{q} \left(\frac{q}{-a + 2\delta} \right)^{N+1} e^{2(-a+\delta)t} \xrightarrow{N \rightarrow \infty} 0,$$

if $q < -a + 2\delta$. At this point we can hence say that

$$|\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| \leq \sum_{n=1}^{\infty} q^n |\mathcal{U}_0^{n+1}(\phi_1 - \phi_2)(t)|, \quad (\text{A.17})$$

which means that we have bounded the term $|\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)|$ from above with another term which we can evaluate thanks to equation (A.13).

With the estimate in (A.13) we can bound from above the term on the right hand side of (A.17) with

$$\sum_{n=1}^{\infty} q^n |\mathcal{U}_0^{n+1}(\phi_1 - \phi_2)(t)| \leq \sum_{n=1}^{\infty} \frac{1}{q} \left(\frac{q}{-(a+\delta)} \right)^{n+1} e^{-(2a+\delta)t} \cdot \|\phi_1 - \phi_2\|_{K_M, t_0}(\delta),$$

which, considering (A.17) gives us

$$|\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t)| \leq C(q, a, \delta) e^{-(2a+\delta)t} \cdot \|\phi_1 - \phi_2\|_{K_M, t_0}(\delta). \quad (\text{A.18})$$

At this point we can hence plug the estimates in (A.18) and (A.13) into (A.16), this gives us the following

$$\begin{aligned} & |-a(\mathcal{U}_0(\rho_0(\phi_1) - \rho_0(\phi_2))(t) + \rho_0(t, \phi_1) - \rho_0(t, \phi_2))| \leq \\ & k_1(q, a, \delta) e^{(-3a+2\delta)t} \|\phi_1 - \phi_2\|_{K_M, t_0}(\delta) + k_2(q, a, \delta) e^{-(2a+\delta)t} \|\phi_1 - \phi_2\|_{K_M, t_0}(\delta) \\ & \leq k(q, a, \delta, q) e^{-(2a+\delta)t} \|\phi_1 - \phi_2\|_{K_M, t_0}(\delta). \end{aligned} \quad (\text{A.19})$$

With the same procedure just performed, we can obtain the following bound

$$|-b(\mathcal{U}_+(\rho_+(\phi_1) - \rho_+(\phi_2))(t) + \rho_+(t, \phi_1) - \rho_+(t, \phi_2))| \leq k(a, \delta, \tilde{q}) e^{-(2a+\delta)t} \|\phi_1 - \phi_2\|_{K_M, t_0}(\delta), \quad (\text{A.20})$$

and

$$\left| \int_{\mathbb{R}} (-a(\mathcal{U}_0(\rho(\phi_1) - \rho(\phi_2))(t, K)) + \rho(t, \phi_1, K) - \rho(t, \phi_2, K)) dK \right| \leq \tilde{k}(a, \delta, q) e^{-(2a+\delta)t} \|\phi_1 - \phi_2\|_{K_M, t_0(\delta)}. \quad (\text{A.21})$$

At this point, considering the bounds (A.19)–(A.21) and the structure of W which is given explicitly in (3.23) we can assert that

$$|W(t, \phi_1) - W(t, \phi_2)| \lesssim e^{-(2a+\delta)t} \cdot \|\phi_1 - \phi_2\|_{K_M, t_0(\delta)}.$$

Concluding the estimate. □

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